UNIVERSITY OF ZAGREB
Faculty of Mechanical Engineering and Naval Architecture

VALIDATION OF RIGID BODY MOTION AND
COMPUTATIONAL FLUID DYNAMICS IN SPECTRAL
SPACE

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Student: Filip Volarić

Zagreb, 2017
I hereby declare that this thesis is entirely the result of my own work except where otherwise indicated. I have fully cited all used sources and I have only used the ones given in the list of references.

I am eminently thankful to Professor Hrvoje Jasak for allowing me to pursue and accomplish this thesis.

I would like to express my sincere gratitude to Inno Gatin for his time, patience and valuable advice. Without his knowledge and experience this thesis would not be possible.

I express my sincere gratitude to my parents and to my sister for all the support they have provided me over the years. Thank you for your patience and for being of such great support.

Finally I would like to thank my friends and colleagues with whom I spent countless hours in a positive working environment on the 8th floor.

Thank you,
Filip volarić
DIPLOMSKI ZADATAK

Student: Filip Volarić  
Mat. br.: 0035190377

Naslov rada na hrvatskom jeziku: Validacija sprege gibanja krutog tijela i računalne dinamike fluida u frekvenčnoj domeni

Naslov rada na engleskom jeziku: Validation of Rigid Body Motion and Computational Fluid Dynamics in Spectral Space

Opis zadatka:
The Harmonic Balance (HB) method in Computational Fluid Dynamics (CFD) approaches the problem of transient quasi-periodic flows via spectral decomposition of a time signal within the Finite Volume (FV) framework. While HB is usually applied to turbulent compressible single phase flows in turbomachinery, it is also applicable to two-phase free surface flows encountered in the field of marine engineering. In predictions of added resistance it is necessary to account for the coupling between fluid flow and motion of the marine object. In order to predict the rigid body motion in conjunction with the HB method, a spectral rigid body motion algorithm is developed. In this study, the candidate shall investigate and validate the spectral rigid body motion algorithm, coupled with the HB fluid flow solution method.

The candidate shall perform the following tasks within this project:
- Perform a literature survey on spectral methods in CFD, including HB and other variants;
- Perform a detailed mathematical investigation of the spectral rigid body motion algorithm;
- Suggest possible improvements of the existing implementation of the spectral rigid body motion algorithm;
- Perform simple two-dimensional simulations of a moving body in two-phase flow with surface waves;
- Perform a validation of the spectral rigid body motion simulation by comparing the results with the transient method;
- Present the comparison of results and computational time for the spectral and transient solution.

The Thesis shall list the bibliography and any assistance received during this study.


Zadatak zadao: 
Prof. dr. sc. Hrvoje Jasak

Predsjednica Povjerenstva: 
Prof. dr. sc. Tanja Jurčević Lulić
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## Nomenclature

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<td>kg</td>
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<td>( N )</td>
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\textbf{Nomenclature} & \\
\hline
$R_{S_l}$ & $l$-th sine Fourier coefficient of $R$ \\
$SE(3)$ & Special Euclidean group \\
$SO(3)$ & Special orthogonal group \\
\textbf{T} & Base period \\
\textbf{$t_k$} & $k$-th discrete time instant \\
\textbf{U} & Mean convective velocity \\
\hline
\end{tabular}
\end{center}
Abbreviations

**SWENSE**  Spectral Wave Explicit Navier Stokes Equation

**CFD**  Computational Fluid Dynamics

**LS**  Level Set

**HB**  Harmonic Balance

**FV**  Finite Volume

**GFM**  Ghost Fluid Method

**FFT**  Fast Fourier Transform

**DFT**  Discrete Fourier Transform
Abstract

Over the last decade the Harmonic Balance (HB) method was developed in the field of Computer Fluid Dynamics (CFD). HB method transforms a transient periodic problem into a set of coupled steady-state problems. Although the HB method was first applied to single-phase flows in the field of turbomachinery, it is also applicable to two-phase free surface flows which can be found in the field of naval hydrodynamics. In the field of naval hydrodynamics it is very important to accurately assess the drag force of the ship in waves during the design process, so it is necessary to couple fluid flow and motion of the ship. Hence, the HB method can only be used in the assessment of drag force if the corresponding spectral rigid body motion algorithm is developed. Therefore, in this thesis HB method is described first, which is used to simulate flow, and then the detailed description of spectral rigid body motion algorithm is given. The new algorithm is validated with two test cases. Results of the first case are compared to an analytical solution, and results of the second case are compared with a transient simulation.

Keywords: CFD, Free Surface Flows, Frequency domain, Harmonic Balance, Rigid Body Dynamics.
Sažetak

U području računalne dinamike fluida se u posljednjih desetak godina razvija metoda harmoničke ravnoteže koja transformira problem tranzijentnog periodičnog strujanja u sustav međusobno ovisnih stacionarnih problema. Iako se metoda harmoničke ravnoteže najviše razvila za jednofazna strujanja u području turbostrojeva, može se primjeniti na simulaciju dvofaznih strujanja sa slobodnom površinom koja se javljaju u području brodske hidrodinamike. Tijekom projektiranja forme broda vrlo je bitno dobro procijeniti silu otpora pri plovidbi u valovitom moru, pa je stoga bitno povezati strujanje fluida i gibanje objekta koje je uzrokovano strujanjem. Budući da je razvijena metoda harmoničke ravnoteže za dvofazna strujanja sa slobodnom površinom, potrebno je razviti komplementarnu metodu za proračun gibanja krutog tijela. U ovom radu prvo je dan opis metode harmoničke ravnoteže koja se koristi za simulaciju dvofaznog strujanja, te je nakon toga dan detaljan izvod algoritma za gibanje tijela u frekvencijskoj domeni. Novi algoritam je testiran sa dva različita slučaja. Rezultati prvog slučaja su uspoređeni sa analitičkim rješenjem, dok su rezultati drugog slučaja uspoređivani sa rješenjem tranzijentne simulacije.

Ključne riječi: Računalna dinamika fluida, Strujanje na slobodnoj površini, Frekvencijska domena, Metoda harmoničke ravnoteže, Dinamika krutog tijela.
Prošireni Sažetak

Uvod

Cilj ovog rada je razviti i validirati metodu za gibanje krutog tijela u frekvencijskoj domeni. Budući da je u računalnoj dinamici fluida već razvijena metoda harmoničke ravnoteže koja preko Fourierove transformacije transformira jedan tranzijentni problem u niz povezanih stacionarnih problema, sljedeći korak je razvoj metode za gibanje krutog tijela koja je kompatibilna sa njom. Budući da je rješavanje strujanja fluida i gibanja krutog tijela povezano, u ovom radu su prvo dane jednadžbe za strujanje fluida, a nakon toga jednadžbe za gibanje tijela. Nadalje će biti prikazani rezultati validacije razvijenog algoritma na dva testna slučaja. Algoritam je razvijen unutar foam-extend softwarea za računalnu dinamiku fluida [1].

Matematički model

Osnovne jednadžbe

Matematički model korišten za modeliranje nestlačivog, turbulentnog, dvofoznog strujanja dva nemješajuća fluida sa "oštrim" granicom sastoji se od jednadžbe očuvanja mase, jednadžbe očuvanja količine gibanja i Level Set jednadžbe kojom se prati pozicija slobodne površine. Te jednadžbe su prvo dane u osnovnom obliku, a zatim će na njih biti primjenjena metoda SWENSE dekompozicije, da bi na kraju osnovne jednadžbe bile dane vremensko-spektralnoj formi.

Granica $\Gamma$ razdvaja dvije faze, u ovom radu vodu i zrak, od kojih svaka ima konstantnu gustoću, $\rho = \rho_v$ u vodi i $\rho = \rho_a$ u zraku.

Uzimajući u obzir konstantnost gustoće po fazama i kontinuiranost polja brzine zbog kinematskog rubnog uvjeta na slobodnoj površini, jednadžba očuvanja mase glasi:

$$\nabla \cdot \mathbf{u} = 0.$$  (1)

Prema Vukčević et al [16] zakon očuvanja količine gibanja za nestlačivo turbulentno dvofozno strujanje glasi:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{u}) - \nabla \cdot (\mathbf{v}_e \nabla \mathbf{u}) = \frac{-1}{\rho} \nabla p_d + \nabla \mathbf{u} \cdot \nabla \mathbf{v}_e,$$  (2)

gdje je $\mathbf{v}_e$ efektivna kinematska viskoznost koja uključuje i turbulentnu viskoznost, $\rho$ je gustoća koja ima diskontinuitet na slobodnoj površini, $p_d$ polje dinamičkog tlaka: $p_d = p - \rho g \cdot \mathbf{x}$, gdje $p$ označava totalni tlak, $g$ gravitacijsko ubrzanje, a $\mathbf{x}$ radij vektor.

Pozicija slobodne površine dobiva se korištenjem Level Set metode koja je pogodna za primjenu uz SWENSE dekompoziciju:
\[
\frac{\partial \psi}{\partial t} + \nabla \cdot (c \psi) - \psi \nabla \cdot \xi - b \nabla \cdot (\nabla \psi) = b \frac{\sqrt{2}}{\varepsilon} \tanh \left( \frac{\psi}{\varepsilon \sqrt{2}} \right).
\]

U jednadžbi (3) \( b \) i \( \varepsilon \) su numerički parametri: \( b \) je koeficijent difuzije, a \( \varepsilon \) je parameter širine razmazivanja. \( \xi \) je modificirana konvektivna brzina:

\[
\xi = u + b \frac{\sqrt{2}}{\varepsilon} \tanh \left( \frac{\psi}{\varepsilon \sqrt{2}} \right) \nabla \psi + b \kappa \frac{\nabla \psi}{|\nabla \psi|},
\]

(4)

Da bi matematički model bio potpun potrebno je navesti i odgovarajuće uvjete skoka. Ghost Fluid metoda (GFM) koja se koristi u ovom radu implicitno uzima u obzir diskontinuitete gradijenta tlaka i gustoće na slobodnoj površini zadovoljavajući kinemske i dinamičke rubne uvjete slobodne površine. Kinematskim rubnim uvjetom slobodne površine propisano je kontinuirano polje brzine na slobodnoj površini odnosno jendakost polja brzine koje se nalazi infinitenzimalno blizu slobodnoj površini u težem fluidu sa poljem brzine u infinitenzimalnoj blizini slobodne površini u lakšem fluidu:\footnote{U svim jednadžbama vezanima uz GFM oznaka + označuje teži fluid, a − lakši.}

\[
[u] = u^− − u^+ = 0.
\]

(5)

Dinamičkim rubnim uvjetom propisana je ravnoteža naprezanja na slobodnoj površini koja se mogu podijeliti na normalnu komponentu koja nastaje zbog tlaka i na tangencijalnu komponentu koja nastaje zbog viskoznih naprezanja. GFM korištena u ovom radu uzima u obzir skok gradijenta tlaka zbog skoka gustoće na slobodnoj površini koji se može izraziti kao:

\[
[\rho] = \rho^− − \rho^+,
\]

(6)

a zanemaruje utjecaje povšinske napetosti. Zanemarenjem povšinske napetosti dobivamo uvjet diskontinuiranog polja tlaka koji se uvođenjem dinamičkog tlaka može zapisati u sljedećem obliku:

\[
[p_d] = −[\rho] \frac{\rho}{\varepsilon} \cdot \mathbf{x}.
\]

(7)

Tangencijalna naprezanja su aproksimirana linearnom interpolacijom kinematske viskoznosti na slobodnoj površini:

\[
\nu_e = \alpha \nu_{e,w} + (1 − \alpha) \nu_{e,a}.
\]

(8)

Linearna interpolacija kinematske viskoznosti opravdana je za "large-scale" površinska strujanja korištena u ovom radu [14].

Također iz jednadžbe 2 zbog kontinuiranog polja brzine \( \mathbf{u} \) i pretpostavljenje kontinuiranosti kinematičke viskoznosti \( \nu_e \) proizlazi dodatni uvjet skoka:

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kojeg je prema mnogim autorima nužno uzeti u obzir [16].

**Osnovne jednadžbe u SWENSE obliku**

SWENSE dekompozicija temelji se na rastavljanju proizvoljnog polja $\xi$ na incidentnu $\xi_I$ i na perturbiranu $\xi_P$ komponentu:

$$\xi = \xi_I + \xi_P.$$  (10)

Svrha SWENSE dekompozicije je opisivanje glavnih značajki valova na slobodnoj površini sa modelom potencijalnog strujanja iz kojeg proizlazi polje $\xi_I$ te naknadno dodavanje nelinearnih, viskoznih i turbulentnih efekata poljem $\xi_P$. Iako je izbor incidentnog polja $\xi_I$ slučajno, pretpostavlja se da ono predstavlja dobru procjenu potpunog rješenja. U nastavku će biti dane osnovne jednadžbe koje su dobivene dekompozicijom jednadžbi (1), (2) i (3) danih u prethodnom poglavlju. Napomena, sva incidentna polja nastala dekompozicijom dobivaju indeks $I$ dok sva perturbirana polja dobivaju indeks $P$.

Dekompozicijom jednadžbe (1) dobiva se jednadžba kontinuiteta u SWENSE obliku:

$$\nabla \cdot u_P = -\nabla \cdot u_I.$$  (11)

iako je polje brzine u potencijalnom strujanju solnoidalno, $\nabla \cdot u = 0$, divergencija incidentnog polja brzine se zadržava jer se sa tim članom se u jednadžbi (11) rješavaju greške kontinuiteta nastale mapiranjem tog polja na diskretiziranu mrežu volumena.

Dekompozicijom jednadžbe (2) dobiva se zakon očuvanja količine gibanja u SWENSE obliku:

$$\frac{\partial u_P}{\partial t} + \nabla \cdot (u_P u_P) - \nabla \cdot (v_e \nabla u_P) =$$
$$- \frac{\partial u_I}{\partial t} - \nabla \cdot (u_I u_I) + \nabla \cdot (v_e \nabla u_I) - \frac{1}{\rho} \nabla p_d + \nabla \cdot (\nabla v_e).$$  (12)

U jednadžbi (12) se može uočiti da nisu sva polja dekompozirana: polje brzine $u$ u konvektivnom članu, dinamički tlak $p_d$, tj. gradijent dinamičkog tlaka te član $\nabla u \cdot \nabla v_e$. Razlog tome leži u različitom tretiranju tih članova u numeričkom postupku. Konvektivna brzina nije dekompozirana zbog linearizacije konvektivnog člana sa explicitnim volumskim protokom iz prethodnog vremenskog koraka, član $\nabla u \cdot \nabla v_e$ nije dekompoziran jer se tretira eksplicitno u numeričkom algoritmu dok se gradijent dinamičkog tlaka tretira u skladu sa GFM metodom pa nije potrebna njegova dekompozicija.

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Level Set jednadžbu u SWENSE obliku dobivamo dekompozicijom jednadžbe (3):

\[
\frac{\partial \psi}{\partial t} + \nabla \cdot (c \psi P) - \psi P \nabla \cdot (\nabla \psi) = 
\frac{\partial \psi}{\partial t} - \nabla \cdot (c \psi_I) + \psi_I \nabla \cdot (\nabla \psi_I) + b \sqrt{\frac{2}{\varepsilon}} \tanh \left( \frac{\psi}{\varepsilon \sqrt{2}} \right). 
\]

(13)

Modificirana konvektivna brzina \(c\) nije podložna dekompoziciji iz istog razloga kao i brzina \(u\) u jednadžbi (12). Također član \(b \sqrt{\frac{2}{\varepsilon}} \tanh \left( \frac{\psi}{\varepsilon \sqrt{2}} \right)\) nije dekompoziran jer se tretira eksplicitno u numeričkom algoritmu.

**Osnovne jednadžbe u HB obliku**

Metoda harmoničke ravnoteže koristi se kod vremenski periodičkih strujanja sa izraženom baznom frekvencijom za transformaciju tranzijentnog problema u niz spregnutih stacionarnih problema. Tranzijentne jednadžbe su transformirane u niz stacionarnih jednadžbi koje su povezane izvorskim članom koji predstavlja vremensku derivaciju u frekvencijskoj domeni. Osnovni uvjet za primjenu metode harmoničke ravnoteže je vremensko osrednjeno strujanje u kojem je moguće svaku varijablu razviti u Fourierov niz sa konačnim brojem harmonika \(N\):

\[
Q(t) = Q_0 + \sum_{l=1}^{N} \left( Q_{C_l} \cos(l \omega t) + Q_{S_l} \sin(l \omega t) \right),
\]

(14)

gdje \(Q\) predstavlja polje u vremenskoj domeni, \(Q\) Fourierov koeficijent tog polja u frekvencijskoj domeni, indeksi \(S_l\) i \(C_l\) predstavljaju sinusni odnosno kosinusni Fourierov koeficijet, a \(\omega\) je poznata bazna frekvencija polja \(Q\).

Standardna transportna jednažba u vremenskoj domeni ima sljedeći oblik:

\[
\frac{\partial Q}{\partial t} + \mathcal{R} = 0,
\]

(15)

gdje je \(\mathcal{R}\) predstavlja konvektivni, difuzijski i izvorski član:

\[
\mathcal{R} = \nabla \cdot (u \mathcal{Q}) - \nabla \cdot (\gamma \nabla \mathcal{Q}) - S_{\mathcal{Q}}.
\]

(16)

Uz razvijanje polja \(\mathcal{Q}\) i \(\mathcal{R}\) u Fourierov niz te primjenom matričnog prikaza diskretnine Fourierove transformacije (DFT), \(Q = \mathcal{F} \mathcal{Q}\), standardna transportna jednažba poprima sljedeći oblik:

\[
\omega \mathcal{F}^{-1} \mathcal{A} \mathcal{F} + \mathcal{R} = 0.
\]

(17)

Jednadžba (17) predstavlja skup kvazi-stacionarnih jednadžbi spojenih elementima izvan glavne dijagonale matrice \(\mathcal{F}^{-1} \mathcal{A} \mathcal{F}\). Usporedbom jednadžbi (15) i (17) primjećuje se da je
vremenska derivacija zamijenjena izvorskim članom:

\[
S(\varphi) = \omega E^{-1} A E \varphi, \\
S_l(\varphi) = -\frac{2\omega}{2N+1} \left( \sum_{k=1}^{2N} P_{k-l} \varphi_k \right), \text{ za } l = 1 \ldots 2N+1, 
\]

gdje je \( S(\varphi) \) vektor izvorskog članova u kojem je \( S_l \) \( l \)-ti element u vektoru. \( P \) je matrica konstanti definirana kao:

\[
P_r = \sum_{m=1}^{N} r \sin(rm\omega\Delta t), \text{ za } r = -N \ldots N, 
\]

za \( \Delta t = T/(2N+1) \). 

Može se vidjeti da se metodom harmoničke ravnoteže tranzijentni problem transformira u skup \( 2N+1 \) kvazi-stacionarnih problema koji se rješavaju u jednako razmaknutim vremenskim trenucima perioda \( T \). Stacionarne jednadžbe su spojene preko izvorskog člana opisanog jednadžbom (18) koji zamjenjuje vremensku derivaciju iz tranzijentne jednadžbe, dok su konvektivni, difuzijski i izvorski članovi iz tranzijentne jednadžbe ostali nepromijenjeni. 

U nastavku će biti dane osnovne jednadžbe u HB obliku.

U jednadžbi kontinuiteta (11) nema člana sa vremenskom derivacijom pa je oblik jednadžbe nepromijenjen, jedna razlika je da se polje brzine sada računa u jednako razmaknutim vremenskim trenucima unutar jednog vremenskog perioda što rezultira sa \( 2N+1 \) međusobno nezavisnih jednadžbi:

\[
\nabla \cdot u_{P,l} = -\nabla \cdot u_{I,l}, \quad l = 1 \ldots 2N+1.
\]

U zakonu očuvanja količine gibanja (12) postoje dva člana sa vremenskom derivacijom koje je potrebno zamijeniti sa HB izvorskim članovima danim jednadžbom (18):

\[
S_l(u_P) + \nabla \cdot (u_l u_P) - \nabla \cdot (v_{e_l} \nabla u_P) = \\
- S_l(u_I) + \nabla \cdot (u_l u_I) + \nabla \cdot (v_{e_l} \nabla u_I) - \frac{1}{\rho_I} \nabla p_{d_l} + \nabla u_I \cdot \nabla V_{e_l}, \quad l = 1 \ldots 2N+1. 
\]

Potrebno je primjetiti da svaka jednadžba ima odgovarajuća polja gustoće \( \rho_l \) i efektivne kinematske viskoznosti \( v_{e_l} \). Budući da su dva HB izvorska člana u jednadžbi (21) linearni operatori moguće ih je staviti u jedan član, međutim to nije napravljeno jer se izvorski član perturbiranog polja brzine \( u_P \) tretira implicitno dok se izvorski član incidentnog polja brzine \( u_I \) tretira eksplicitno.

Level Set jednadžba (13) kao i zakon očuvanja količine gibanja ima dva člana sa
vremenskim derivacijama koja je potrebno zamijeniti sa HB izvornim članom daniм jednadžbom (18):

\[ S_l(\psi_P) + \nabla \cdot (c_l \psi_P) - \psi_R \nabla \cdot \psi_l - b \nabla \cdot (\nabla \psi_R) = \]

\[ - S_l(\psi_I) - \nabla \cdot (c_l \psi_I) + \psi_I \nabla \cdot \psi_I + b \sqrt{2} \ \text{tanh} \left( \frac{\psi_I}{\varepsilon \sqrt{2}} \right). \]  

(22)

Isto kao i u zakonu očuvanja količine gibanja dva HB izvorska člana je moguće spojiti u jedan, međutim to nije napravljeno radi implicitnog tretiranja perturbacijskog polja.

**Dinamika krutog tijela**

U dinamici "multibody" sustava postoje razne klase problema. Jedna od tih klasa je izravna dinamika koja se bavi određivanjem gibanja sustava podvrgnutog unaprijed prepisanim silama i momentima. Iako se mehanički sustavi inače sastoje od više međusobno povezanih krutih tijela u ovom radu se taj sustav sastoji od samo jednog krutog tijela. Budući da se naš sustav sastoji od samo jednog krutog tijela njegovu konfiguraciju je moguće odrediti rješavanjem dvije obične diferencijalne jednadžbe. Konfiguracija tijela je poznata kada je poznata njegova pozicija i rotacija. Prva jednadžba opisuje gibanje krutog tijela prilikom translacije i naziva se Newtonova jednadžba dok druga jednadžba koja se naziva Eulerova jednadžba opisuje gibanje krutog tijela prilikom rotacije. Generalno gibanje krutog tijela se može predočiti kao transformacija koordinatnih sustava; translacija koordinatnog sustava vezanog za tijelo u odnosu na inercijalni koordinatni sustav te rotacija koordinatnog sustava vezanog za tijelo. Uz ispunjavanje određenih uvjeta prilikom izbora lokalnog koordinatnog sustava, Newtonova i Eulerova jednadžba su raspregnute.

Promjenu položaja čestice pratimo pomoću vektora položaja \( \mathbf{x} \) koji povezuje ishodište referentnog inercijalnog koordinatnog sustava i trenutni položaj čestice tj. centra masa krutog tijela. Osnovna jednadžba kojom se određuje translatorno gibanje tijela je Newtonova jednadžba:

\[ ma = f_R, \]  

(23)

gdje je \( m \) masa tijela, \( a \) ubrzanje centra masa krutog tijela, a \( f_R \) rezultantna sila koja djeluje u centru masa.

Radi uključenja efekta opruge i viskoznog prigušivača jednadžbu (23) je potrebno modificirati. Izuzimanjem sile u opruzi i sile u prigušivaču iz rezultantne sile te uzimanjem u obzir smjer djelovanja tih sila jednadžba (23) poprima sljedeći oblik:

\[ m\ddot{x} + k\dot{x} + \xi x = f. \]  

(24)
gdje je $m$ dijagonalna matrica sa iznosima mase na dijagonali, $\zeta$ je dijagonalna matrica koeficijenata krutosti, a $k$ je dijagonalna matrica koeficijenata viskoznog prigušenja. Nadalje, $\mathbf{x}$ je vektor položaja centra masa krutog tijela, $\dot{\mathbf{x}}$ je vektor relativne brzine, a $\ddot{\mathbf{x}}$ je vektor akceleracije.

Da bi mogli gibanje tijela rješavati u frekvencijskoj domeni potrebno je sve vremenski ovisne varijable u jednadžbi (24) razviti u Fourierov niz. Ovdje se koristi razvoj u konačni kompleksni Fourierov niz. Općenitu varijablu $q(t)$ razvijamo u kompleksni Fourierov niz prema:

$$q(t) = Q_0 + \sum_{p=1}^{N} Q_p e^{i\omega pt},$$  \hspace{1cm} (25)

gdje je $i$ imaginarna jedinica $i = \sqrt{-1}$, $Q_p$ kompleksna amplituda $p$-tog harmonika, a $\omega$ bazna frekvencija. Kao što se može vidjeti u jednadžbi (24) postoje i derivacije vremenske varijable pa je potrebno derivirati Fourierov niz (jednadžba 25). Prva derivacija je definirana kao:

$$\dot{q}(t) = \sum_{p=1}^{N} ip\omega Q_p e^{i\omega pt},$$  \hspace{1cm} (26)

dok je druga derivacija definirana kao:

$$\ddot{q}(t) = \sum_{p=1}^{N} -p^2 \omega^2 Q_p e^{i\omega pt}.$$  \hspace{1cm} (27)

Razvojem u kompleksni Fourierov red pomaka $\mathbf{x}$ i njegovih derivacija te njihovim uvrštanjem u jednadžbi (24) dobivamo jednadžbu translatorskog gibanja krutog tijela u frekvencijskoj domeni. Sređivanjem te grupiranjem članova uz isti harmonik dobiva se izraz za izračun Fourierovih koeficijenata pomaka krutog tijela u frekvencijskoj domeni:

$$X_{p_j} = \frac{F_{p_j}}{-mp^2\omega^2 + ip\omega k_j + c_j}, \quad p = 1 \ldots N \text{ i } j = x, y, z,$$  \hspace{1cm} (28)

gdje je $X_{p_j}$ vektor Fourierovih koeficijenata pomaka u smjeru koordinatnih osi $p$-tog harmonika.

Nakon izračuna Fourierovih koeficijenata pomaka moguće je na temelju definicije brzine $\dot{u}$ kao trenutne promjene vektora položaja:

$$\dot{u} = \frac{dx}{dt},$$  \hspace{1cm} (29)

te primjene kompleksnog Fourierovog niza dobiti izraz za određivanje Fourierovih koeficijenata brzine:

$$U_{p_j} = ip\omega X_{p_j} \text{ za } p = 1 \ldots N,$$  \hspace{1cm} (30)

gdje je $U_{p_j}$ vektor Fourierovih koeficijenata brzine u smjeru koordinatnih osi $p$-tog harmonika.

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Nakon određivanja pozicije tijela potrebno je odrediti i njegovu orijentaciju da bi u potpunosti znali konfiguraciju tijela. Orijentacija tijela je jednoznačno određena matricom rotacije $R \in SO(3)$. $SO(3)$ grupa kojoj pripada matrica rotacije je također i Liejeva grupa pa se u skladu sa radom Müllera i Terzea [15] matrica rotacija može računati kao:

$$ R = I + \frac{\sin \|\xi\|}{\|\xi\|} \xi + \frac{1 - \cos \|\xi\|}{\|\xi\|^2} \xi \xi, $$

gdje je $I$ jedinična matrica, a $\xi$ je moguće odrediti iz kutne brzine $\omega$:

$$ \dot{\xi} = \omega. $$

Jednadžba (32) nam osigurava drugi red točnosti.

Iz jednadžbe (32) je vidljivo da je za određivanje vektora $\xi$ a time i matrice rotacije $R$ potrebno odrediti kutnu brzinu $\omega$. Kutnu brzinu određujemo iz Eulerove jednadžbe koja opisuje rotacijsko gibanje krutog tijela:

$$ J \dot{\omega} + \omega \times J \omega = l, $$

gdje je $J$ tenzor inercije krutog tijela, $\omega$ je kutna brzina, a $l$ rezultantni moment koji djeluje na tijelo.

U ovom radu gibanje se rješava u koordinatnom sustavu vezanom za tijelo. Koordinatni sustav vezan za tijelo ima ishodište u centru masa tijela jer su u tom slučaju raspregnute jednadžbe za translaciju i rotaciju. Također budući da je u ovom rješavanju tenzor inercije $J$ definiran kao dijagonalan tenzor lokalni koordinatni sustav mora biti orijentiran prema glavnim osima inercije. Jednadžba (33) zapisana u matričnoj formulaciji glasi:

$$ I \hat{\omega} + \hat{\omega} I \omega = l, $$

gdje je $\hat{\omega}$ antisimetrična matrica oblika:

$$ \hat{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} $$

koja se koristi za izračun vektorskog produkta.

Analognog Newtonovoj jednadžbi i Eulerovu jednadžbu (34) je potrebno zapisati u frekvencijskoj domeni. Primjenom raspisa kutne brzine $\omega$ i njene derivacije $\hat{\omega}$ u kompleksni Fourierov niz prema jednadžbama (26) i (27) te naknadnog sređivanja, Eulerova jednadžba po komponentama poprima sljedeću formu u frekvencijskoj domeni:
\[ \Omega_{px} = \frac{1}{ip\omega J_{xx}} \left[ L_{px} + (J_{yy} - J_{zz}) \sum_{k=0}^{p} \Omega_{px} \Omega_{(p-k)z} \right], \]
\[ \Omega_{py} = \frac{1}{ip\omega J_{yy}} \left[ L_{py} + (J_{zz} - J_{xx}) \sum_{k=0}^{p} \Omega_{py} \Omega_{(p-k)x} \right], \text{ za } p = 1 \ldots N \] (36)
\[ \Omega_{pz} = \frac{1}{ip\omega J_{zz}} \left[ L_{pz} + (J_{xx} - J_{yy}) \sum_{k=0}^{p} \Omega_{pz} \Omega_{(p-k)y} \right], \]

gdje su \( \Omega_{px}, \Omega_{py} \) i \( \Omega_{pz} \) kompleksni Fourierovi koeficijenti \( p \)-tog harmonika u smjeru osi \( x, y \) i \( z \) od kutne brzine \( \omega \), a \( L_{px}, L_{py} \) i \( L_{pz} \) kompleksni Fourierovi koeficijenti \( ptog \) harmonika u smjeru osi \( x, y \) i \( z \) od momenta \( l \).

Iako se matrica rotacije \( R \) izračunava u vremenskoj domeni vektor \( \vec{\xi} \) koji je nužan za izračun matrice rotacije dobiva se inverznom Fourierovom transformacijom njegovih Fourierovih koeficijenata iz spektralne domene. Primjenom razvoja varijabli iz jednadžbe (32) u Fourierov niz dobiva se:
\[ \Xi_{pj} = \frac{\Omega_{pj}}{ip\omega}, \text{ za } p = 1 \ldots N \text{ i } j = x, y, z, \] (37)
gdje je \( \Xi_{pj} \) Fourierov koeficijent \( p \)-tog harmonika u smjeru osi \( j \) od vektora \( \vec{\xi} \).

Kruto tijelo mijenja svoju konfiguraciju zbog sila i momenata koji djeluju na njega. Sila koja dijeluje na tijelo koje se nalazi u fluidu nastaje zbog dva uzroka: dio sile nastaje zbog tlaka \( f_p \) i dio sile nastaje zbog viskoznih efekata \( f_v \):
\[ \vec{f} = \vec{f}_p + \vec{f}_v. \] (38)

Analogno sili, moment se isto dijeli na dio koji nastaje zbog tlaka i dio koji nastaje zbog viskoznih efekata:
\[ \vec{l} = \vec{l}_p + \vec{l}_v. \] (39)

Sile i momenti u vremenskoj domeni se izračunavaju u globalnom koordinatnom sustavu kao:
\[ \vec{f}_p = \sum_{bf} \xi_f P_f + mg, \] (40)
\[ \vec{f}_v = \sum_{bf} \rho_f \nu_e f \xi_f \cdot T^*, \] (41)
\[ \vec{l}_p = \sum_{bf} \xi_f \times \xi_f P_f, \] (42)
\[ f_v = \sum_{bf} f_f \times \left( \rho_{f,v} \xi_{f,f} \cdot \mathbf{T}^* \right), \]  \hspace{1cm} (43)

gdje \( \sum \) označava sumaciju po svih plohama tijela, \( \rho_f \) je odgovarajuća gustoća na rubnoj plohi, a \( \nu_{e,f} \) je efektivna kinematska viskoznost. \( \mathbf{T}^* \) je devijatorski dio tenzora naprezanja \( \mathbf{T} \) koji je definiran kao dvostruko simetrični dio \( \nabla \mathbf{u} \) tenzora. Vektor \( r_f \) je vektor položaja koji povezuje trenutnu rubnu plohu i centar masa krutog tijela.

Budući da se polje strujanja rješava sa HB metodom, sva polja strujanja su izračunata u \( 2N + 1 \) jednako razmaknutih trenutaka perioda \( T \). Stoga su sile i momenti, za čije su izračunavanje potrebna polja tlaka \( p \) i brzine \( u \), također poznati u diskretnim vremenskim trenucima. Diskrete vrijednosti sile i momenta se transformiraju u frekvencijsku domenu preko diskretnog Fourierovih transformacija.

**Validacija spektralnog pristupa gibanju krutog tijela**

**Validacija gibanja uzrokovanog konstantnim silama i momentima**

Novi algoritam za izračun gibanja krutog tijela u frekvencijskoj domeni je validiran sa dva testna slučaja. U prvom slučaju se rezultati uspoređuju s analitičkim rješenjem. Da bi mogli odrediti analitičko rješenje sile se postavljaju na konstantnu vrijednost amplitude prilikom simulacije umjesto da se izračunavaju iz polja strujanja. Nadalje, pravokutno tijelo je postavljeno u jednofazno, laminarno i neviskozno strujanje tako da se efekti prigušenja mogu zanemariti. Provedeno je 9 simulacija za validaciju translacionog gibanja, te također 9 simulacija za validaciju rotacijskog gibanja.  

Simulacije se razlikuju po iznosima amplituda sila/momenata, te po korištenom redu amplituda sile odnosno momenta.

Dimenzije računalne domene korištene kod simulacija za validaciju translacionog gibanja prikazane su na slici 1, dok su dimenzije domene za validaciju rotacijskog gibanja dane na slici 2.

![Slika 1: Računalna domena kod validacije translacijskog gibanja](image-url)
**Tablica 1: Sila korištene za validaciju rješenja translacionog gibanja**

<table>
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Kao što je već rečeno validacija translacijskog gibanja je provedena sa 9 simulacija. Vrijednosti korištenih sila se mogu vidjeti u Tablici 1.
Kao što se može vidjeti iz Tablice 1 složenost korištene sile se povećava sa povećanjem broja simulacija. Cilj je bio pokrivanje širokog raspona mogućih sila. Zadnja simulacija je provedena sa silom koja se sastoji od sva četiri reda sile, ukoliko je njezino poklapanje sa analitičkim rješenjem zadovoljavajuće pretpostavlja se da algoritam radi i za sile viših redova.

**Tablica 2: Rezultati simulacija za validaciju translacijskog rješenja**

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Iz rezultata u tablici 2 može se vidjeti da numerički rezultati za translacijsko gibanje...
Filip Volaric

Prošireni sažetak

konvergiraju u analitičko rješenje. Konvergencija Fourierovih koeficijenata za poziciju \( X_p \) i brzinu \( U_p \) je prikazana na grafovima koje se nalaze u dodatku A.1, dok je usporedba numeričkog i analitičkog rješenja za poziciju \( y \) i brzinu \( u_y \) u vremenskoj domeni dana na grafovima u dodatku A.2.

Momenti koji se koriste za validaciju numeričkog rješenja za rotacijsko gibanje krutog tijela su prikazani u tablici 3. Da bi se izbjegla zabuna između translatorskih i rotacijskih simulacija, rotacijske simulacije se označene rasponom brojeva od 10 do 18.

**Tablica 3: Momenti korišteni za validaciju rješenja rotacijskog gibanja**

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Rezultati simulacija za validaciju rotacijskog gibanja su dani u tablici 4. Iz tih rezultata je vidljivo da numerička rješenja za Fourierove koeficijente Eulerovog kutova \( \Theta_p \) i kutne brzine \( \Omega_p \) konvergiraju u analitička rješenja. Konvergencija Fourierovih koeficijenata je prikazana na grafovima u dodatku B.1. Poklapanje numeričkog i analitičkog rješenja za Eulerov kut \( \theta_c \) i kutnu brzinu \( \omega_c \) u vremenskoj domeni je dano na grafovima u dodatku B.2.
Tablica 4: Rezultati simulacija za validaciju rotacijskog rješenja

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Iz prikazanih rezultata se može zaključiti da algoritam za izračun gibanja u frekvencijskoj domeni daje rezultate identične analitičkom rješenju u slučaju da se sile/momenti ne izračunavaju iz polja strujanja. Naravno potrebno je validirati gibanje u slučaju kada se sile/momenti koji uzrokuju gibanje izračunavaju iz polja strujanja. Stoga su u nastavku dani rezultati drugog testnog slučaja u kojem se validira translatorsko gibanje uzrokovanog utjecajem
površinskih valova.

Validacija translatorskog gibanja uzrokovalog utjecajem valova

U drugom testnom slučaju se rezultati HB simulacije uspoređuju sa rezultatima odgovarajuće tranzijentne simulacije. Dimenzije računalne domene korištene u HB simulaciji su prikazane na slici 3, dok se pojedinosti geometrije potopljenog tijela mogu vidjeti na slici 4.

![Slika 3: Računalna domena za HB simulaciju.](image1)

![Slika 4: Dimenzije krutog tijela.](image2)

Površinski valovi koji uzrokuju gibanje tijela imaju valnu duljinu od $\lambda = 11.84$ m. Iz te vrijednosti valne duljine određuje se kružna frekvencija i temeljni period oscilacije. Međutim potrebno je uzeti u obzir i činjenicu da se tijelo giblje brzinom od $U = 2$ m/s. Stoga se u simulacijama koristi tzv. "encounter" kružna frekvencija koja u sebi sadrži i utjecaj gibanja tijela. "Encounter" kružna frekvencija u ovom slučaju je $\omega_e = 3.343$ rad/s što odgovara "encounter" periodu $T_e = 1.8795$ s. Potrebno je još spomenuti da je visina vala $H = 0.196$ m.

HB simulacija je izvršena sa 4 i 6 harmonika. U tablici 5 su prikazani rezultati HB simulacija kao i relativna razlika u odnosu na rezultate tranzijentne simulacije. Relativna razlika se izračunava kao $\varepsilon = (S_r - S_{hb})/S_r$, gdje $S_r$ označava tranzijentno rješenje, a $S_{hb}$ označava HB rješenje.
Iz tablice se može iščitati da je relativna razlika za prvi red amplitude pomaka 13.6% za simulaciju sa četiri harmonika, dok ta razlika kod simulacije sa šest harmonika padne na 12.23%.

Sile prvog i drugog reda ne konvergiraju u potpunosti u HB simulaciji tj. zadržava se periodična promjena između minimalne i maksimalne vrijednosti. U simulaciji sa četiri harmonika, najveća relativna razlika za prvi red sile iznosi 15.96%, dok u slučaju simulacije sa šest harmonika ta razlika padne na 13.16%. Rezultati za drugi red sile u obje simulacije imaju odstupanje od ±50% od iznosa svoje srednje vrijednosti te se stoga ne mogu uzeti u obzir. Međutim, budući da je iznos drugog reda sile maksimalno ≈ 5% iznosa sile prvog reda, on ne utječe na ukupno gibanje krutog tijela. U tablici 5 se ne nalaze pomaci drugog reda budući da su njihovi iznosi u obje HB simulacije te u tranzijentnoj simulaciji zanemarivi. Konvergencija pomaka prikazana je na slici 5, dok je konvergencija sile prikazana na slici 6.

![Graph](image-url)
Slika 6: Konvergencija prvog reda sile.

Sa slika koje pokazuju polje perturbirane brzine (slike 7a i 7b) u obje simulacije može se zaključiti da su ona slična u prvoj fazi tj. vodi. Međutim, veličina perturbiranog polja brzine u zraku je veća u tranzijntnoj simulaciji, dok je periodičnost polja bolje izražena u HB simulaciji. Razlika u poljima perturbirane brzine je povezana sa postojanjem velikih iznosa tlakova u relaksacijskoj zoni tranzijntne simulacije (slike 8a i 8b). Usprkos tome sa slika 8a i 8b koje prikazuju polje dinamičkog tlaka u obje simulacije za 6 vremenski trenutak se može zaključiti da je polje tlaka slično u okolici potopljenog tijela. Budući da je u ovom slučaju tijelo potopljeno razlike u perturbiranim poljima ne utječu na gibanje.
Zaključak

Iz provedene validacije algoritma za izračun gibanja u frekvencijskoj domeni se vidi da je relativna razlika između rezultata HB i tranzijentne simulacije između 10% i 15%. Takve razlike su prihvatljive budući da je gibanje potopljenog tijela ekstremno blago. Vrijednost prvog reda sile iznosi samo 0.32% težine tijela, dok je amplituda gibanja manja od 10% visine površinskog vala.

U budućem radu nužno je validirati rotacijsko gibanje krutog tijela te provesti 3D simulaciju gibanja sa 6 stupnjeva slobode. Rezultati provedenih simulacija potvrđuju da se HB metoda može koristiti sa simulaciju gibanja tijela pri plovidbi po valovitom moru.
1 Introduction

This work presents the development and validation of time-spectral method for simulation of the wave-induced rigid body motion. Simulation of fully nonlinear, viscous, temporally periodic, large-scale two-phase flows is based on an existing two-phase numerical model developed within a FV Computational Fluid Dynamics (CFD) software foam-extend [1]. This method is intended for general two-phase flows with temporally periodic nature of the boundary conditions. With the temporally periodic boundary conditions, flow field in the domain of the interest is periodic. Periodic flow fields are encountered in the problems with surface waves which are important in the field of naval, offshore and ocean hydrodynamics.

One of the important tasks of naval hydrodynamics is to reliably assess drag force of ships in waves. Reliable assessment of drag force in waves is important in the process of the ship hull optimization which is used to lower fuel consumption. With the strong development of computer technology, we are able to perform realistic CFD simulations. Therefore, for the assessment of drag force viscous CFD methods are used since they offer reliable results, as shown in numerous validation publications [2, 3, 4, 5, 6].

Most of the viscous naval hydrodynamics CFD calculations are performed in time domain. Although frequency domain methods are used in naval hydrodynamics for several decades [7], modern spectral CFD method that could be used in naval hydrodynamics was developed in recent years [9, 10, 11, 12, 13, 8]. It should be noted that the mentioned frequency domain methods which were used in naval hydrodynamics are based on potential flow assumption where the nonlinear free surface boundary conditions are often linearised. Hence, they cannot describe nonlinear flows accurately and therefore cannot be used to assess ship resistance in waves.

Spectral CFD methods were first developed in the field of turbomachinery. In the following text, a brief overview of the publications related to the use of spectral methods for the turbomachinery applications will be presented. He and Ning [9] presented a nonlinear harmonic method applied to Navier-Stokes equations for simulation of unsteady viscous flow around turbomachinery blades. Maple et al [10] developed an adaptive method where the number of resolved harmonics is varied in the domain depending on the required spectral resolution, applied to a supersonic/subsonic diverging nozzle. McMullen and Jameson [11] investigated acceleration techniques for solving the coupled set of steady state problems. Ekici et al [12] used Euler equations based spectral method to simulate helicopter rotor blade flow, while Guédeney et al [13] extended the method for turbomachinery flows with multiple frequencies in order to capture rotor-stator interaction effects.

The nonlinear harmonic method, also called HB, transforms a periodic transient problem into a set of coupled steady-state problems. Transformation of a transient problem in the set of coupled steady-state problems is possible for the flows with temporally periodic nature with a known dominant frequency of oscillation. Since steady state problems are generally faster to
perform than transient problems, HB method should accelerate the calculation. Furthermore, with the use of the HB method the main drawback of the transient simulation of the periodic flow is avoided. Namely when the periodic flow is simulated with transient time-marching techniques a large number of simulated periods is required before fully developed periodical flow can be reached. In this work, implicit coupling of steady state equations in the block matrix is used [14].

Another advantage of the HB method is that steady-state equations provide the ability of automatic optimisation. In naval hydrodynamics, automatic optimisation of hull geometry with respect to wave related objects, such as added wave resistance or ship motion could be achieved.

Simulation of the flow presents only a part of this work. Flow field obtained with HB simulation is then used to calculate the corresponding ship motion. Ship motion is governed by Newton-Euler equations. Newton-Euler equations are usually solved in time domain, however since the flow field is solved in frequency domain, the motion should be also solved in frequency domain. Transformation of Newton-Euler equations in the frequency domain is achieved using Discrete Fourier Transform (DFT). Definition of the rotation matrix is usually problematic since the most widely used parametrisations (Euler angles, quaternions) have some restrictions. In this thesis, parametrisation problems are avoided by calculating the rotation matrix directly via Lie group approach [15].

This thesis is divided into five chapters. The second chapter called Mathematical model presents the governing equations in SWENSE decomposed form as described by Vukčević et al [16]. Next, the HB treatment of time derivative term is described. Furthermore, von Neumann stability analysis of the implicitly coupled HB source term is conducted. The third chapter is used to describe rigid body motion. The chapter is divided into three section: the first section describes translational motion in both time and the frequency domain, while the second section is used to describe rotational motion in both domains. In the third section, calculation of forces and torques is described. In the fourth chapter, translational and rotational motion is validated with the analytical solution for heave force and pitch torque, respectively. The fifth chapter is used to describe a process of heave translation validation on a submerged rigid body. HB simulation results are compared with the corresponding transient simulation. Finally, a conclusion is given.
2 Mathematical model

2.1 Introduction

In this chapter a mathematical model used for description of two-phase, nonlinear and viscous flow is given. First, the governing equations will be presented [16] with the usage of Ghost Fluid Method (GFM) [17] to formulate dynamic pressure and density jump at the interface. Next, the governing equations in SWENSE decomposed form are presented, [16]. Furthermore, a description of governing equations in the time-spectral HB form is given [14].

2.2 Governing equations

In this section the mathematical model for incompressible, turbulent, two-phase flow of two immiscible fluids with a sharp interface is presented. Interface $\Gamma$ separates the two phases, in this thesis water and air, each of which has a constant density, $\rho = \rho_w$ and $\rho = \rho_a$, respectively.

Taking into account piece-wise constant density and continuous velocity field due to the kinematic boundary condition, the continuity equation reads:

$$\nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u}$ is the velocity field. As a general note, in this thesis vectors and vector fields are denoted with one line under the symbol and matrices with two lines.

According to Vukčević et al [17], momentum equation for incompressible, turbulent, two-phase flow reads:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{u}) - \nabla \cdot (\nu_e \nabla \mathbf{u}) = - \frac{1}{\rho} \nabla p_d + \nabla \mathbf{u} \cdot \nabla \nu_e,$$

where $\nu_e$ stands for the effective kinematic viscosity, $\rho$ is the density which has a discontinuity at the interface, $p_d$ denotes the dynamic pressure, $p_d = p - \rho g \cdot x$, where $p$ is the total pressure, $g$ represents constant gravitational acceleration, and $x$ is the radii vector.

The interface is captured with the Level Set (LS) method derived from Phase Field equation. In the Level Set equation derived from Phase Field equation [18], the signed distance function is unbounded, which is suitable for SWENSE decomposition, Additionally, the need for the redistancing algorithm is eliminated [16]. Transport equation for the Level Set field $\psi$ reads:

$$\frac{\partial \psi}{\partial t} + \nabla \cdot (c \psi) - \psi \nabla \cdot \mathbf{c} - b \nabla \cdot (\nabla \psi) = b \frac{\sqrt{2}}{\varepsilon} \tanh \left( \frac{\psi}{\varepsilon \sqrt{2}} \right),$$

where $\mathbf{c}$ is the modified convective velocity defined as:

\[\mathbf{c} = \mathbf{u} - \nabla \cdot \mathbf{u} \nabla \psi \quad \text{with} \quad \mathbf{u} = \frac{\nabla \psi}{\left| \nabla \psi \right|} \times \nabla \psi \]
\[ c = u + b \frac{\sqrt{2}}{\varepsilon} \tanh \left( \frac{\psi}{\varepsilon \sqrt{2}} \right) \nabla \psi + b \kappa \frac{\nabla \psi}{|\nabla \psi|}, \]  

(2.4)

where \( \kappa \) denotes mean interface curvature.

For the mathematical model to be complete, it is necessary to state the appropriate jump conditions at the interface. GFM used in this model implicitly takes into account the discontinuities of pressure gradient and density on the free surface by satisfying the kinematic and dynamic free surface boundary conditions \[17\]. The kinematic free surface boundary condition states that velocity is continuous across the interface, i.e. velocity field infinitesimally close to the interface in the heavier fluid is equal to the velocity field infinitesimally close to the interface in the lighter fluid\(^2\):

\[ [u] = u^- - u^+ = 0. \]  

(2.5)

The dynamic boundary condition states that the stresses at the interface must be in equilibrium. Stress is usually divided into a normal component due to pressure and a tangential component which originates from viscous stress. In this work only pressure gradient jump due to density jump at the interface is taken into account:

\[ [\rho] = \rho^- - \rho^+, \]  

(2.6)

while surface tension effects are neglected. By neglecting surface tension effects, we obtain the condition of discontinuous pressure field that can be written in terms of dynamic pressure in the following form:

\[ [p_d] = -[\rho] g \cdot x. \]  

(2.7)

Tangential stress is approximated by linear interpolation of kinematic viscosity at the interface:

\[ \nu_e = \nu_{e,w} + (1 - \alpha) \nu_{e,a}. \]  

(2.8)

Linear interpolation of kinematic viscosity is justified for large scale free surface flows \[14\] considered in this work.

Additional jump condition that needs to be taken into account comes from the continuous velocity field \( u \) and assumed continuity of the kinematic viscosity \( \nu_e \). If we transfer the term \( \nabla u \cdot \nabla \nu_e \) from RHS to the LHS of Eq. (2.2) we can see that then LHS is continuous and because of that RHS also must be continuous i.e.:

\(^2\)In all equations related to the GFM \( ^+ \) denotes heavier fluid and \( ^- \) denotes lighter fluid.
Eq. (2.9) represents an additional jump condition that must be taken into account along with the conditions stated by Eq. (2.5), Eq. (2.6) and Eq. (2.7). For more details regarding derivation and implementation of GFM the reader is referred to [17].

### 2.3 SWENSE decomposition

Spectral Wave Explicit Navier Stokes Equation (SWENSE) decomposition is based on decomposition of the arbitrary field $\xi$ into incident $\xi_I$ and diffracted (perturbation) $\xi_P$ component [16]:

$$\xi = \xi_I + \xi_P.$$  
(2.10)

The purpose of SWENSE decomposition is to describe the main features of the free surface wave with a potential flow model from which the field $\xi_I$ is calculated. After the incident field $\xi_I$ is obtained, nonlinear, viscous and turbulent effects are superimposed via diffracted field $\xi_P$. Although the choice of the incident field $\xi_I$ is arbitrary, it is assumed that it represents a reasonable estimate of the complete solution.

Governing equations in SWENSE decomposed form are presented below. Note that all incident fields generated by decomposition are denoted with index $I$, while all perturbed fields are denoted with index $P$.

**SWENSE decomposed form of continuity equation (2.1) reads:**

$$\nabla \cdot u_P = -\nabla \cdot u_I.$$  
(2.11)

Although in the potential flow velocity field is solenoidal, $\nabla \cdot u_I = 0$, the term $\nabla \cdot u_I = 0$ is retained in Eq. (2.11). With that term Eq. (2.11) solves continuity errors generated by mapping the field $u_I$ generated from the potential flow model to the discretised volume mesh.

Momentum equation (2.2) in SWENSE decomposed form reads:

$$\frac{\partial u_P}{\partial t} + \nabla \cdot (u_P u_P) - \nabla \cdot (\nu e \nabla u_P) =$$

$$- \frac{\partial u_I}{\partial t} - \nabla \cdot (u_I u_I) + \nabla \cdot (\nu e \nabla u_I) - \frac{1}{\rho} \nabla p_d + \nabla u \cdot \nabla e.$$  
(2.12)

By comparing Eq. (2.2) and Eq. (2.12), it can be seen that only the velocity field $u$ in the time derivative, convection and diffusion terms is decomposed. Decomposition of the convecting velocity field is unnecessary because it is linearised with explicit volume flux from previous time step, dynamic pressure field is not decomposed because it is treated by GFM, [17], and
the velocity in the last term on right-hand side is not decomposed since it is treated explicitly.

Level Set equation (2.3) in SWENSE decomposed form reads:

\[
\frac{\partial \psi_P}{\partial t} + \nabla \cdot (c \psi_P \nabla \psi) - \psi_P \nabla \cdot \mathbf{c} - b \nabla \cdot (\nabla \psi_P) = -\frac{\partial \psi_I}{\partial t} + \nabla \cdot (c \psi_I \nabla \psi_I) + \psi_I \nabla \cdot \mathbf{c} + b \nabla \cdot (\nabla \psi_I) + b \sqrt{2} \tanh \left( \frac{\psi}{\varepsilon \sqrt{2}} \right). \tag{2.13}
\]

Modified convective velocity field \( \mathbf{c} \) is treated in the same way as the convective velocity field in the momentum equation and therefore it is not decomposed. Furthermore, since Level Set field \( \psi \) is treated explicitly in the last term on the right-hand side, it is not decomposed.

After the decomposition of governing fluid flow equations, an overview of the Harmonic Balance method will be given in the next section.

### 2.4 Harmonic balance method

The Harmonic Balance method [8, 14] is used in temporally periodic flows with a known dominant frequency to transform a periodic transient problem into a set of coupled steady-state problems. Transient equations are transformed into a set of steady-state equation coupled via source term that represents the time derivative in the frequency domain. The primary condition that needs to be fulfilled for the usage of Harmonic Balance method is that the flow is temporally periodic in which case every variable can be expanded in a Fourier series with a finite number of harmonics \( N \):

\[
\mathcal{Q}(t) = Q_0 + \sum_{l=1}^{N} \left( Q_{Cl} \cos(l \omega t) + Q_{Sl} \sin(l \omega t) \right), \tag{2.14}
\]

where \( \mathcal{Q} \) denotes general field variable in time, while \( Q \) stands for Fourier coefficient of that variable in frequency domain. \( \omega \) is known dominant i.e. base frequency of \( \mathcal{Q} \) and indices \( S_l \) and \( C_l \) stand for sine and cosine Fourier coefficient, respectively.

Standard transport equation for variable \( \mathcal{Q} \) in time domain has the following form:

\[
\frac{\partial \mathcal{Q}}{\partial t} + \mathcal{R} = 0, \tag{2.15}
\]

where \( \mathcal{R} \) is a condensed way of writing convective, diffusive and source terms:

\[
\mathcal{R} = \nabla \cdot (u \mathcal{Q}) - \nabla \cdot (\gamma \nabla \mathcal{Q}) - S_{\mathcal{Q}}, \tag{2.16}
\]

used for the sake of clarity. \( \mathcal{R} \) is expanded into Fourier series analogous to the Eq. (2.14), with \( Q \) substituted by \( R \). Insertion of \( \mathcal{Q} \) and \( \mathcal{R} \) into the standard transport equation, Eq. (2.15) yields:
\[ R_0 + \sum_{l=1}^{N} l \omega (Q_S \cos(l \omega t) - Q_C \sin(l \omega t)) + \sum_{l=1}^{N} (-R_C \cos(l \omega t) - R_S \sin(l \omega t)) + R_0 = 0. \]  
(2.17)

After equating the terms with the same harmonic in Eq. (2.17), 2N + 1 equations are obtained. 
N equations for sine part:

\[ -l \omega Q_C + R_S = 0, \quad \text{for} \quad l = 1 \ldots N, \]  
(2.18)

one equation for the mean value:

\[ R_0 = 0, \]  
(2.19)

and N equations for cosine part:

\[ l \omega Q_S + R_C = 0, \quad \text{for} \quad l = 1 \ldots N. \]  
(2.20)

Equations (2.18), (2.19) and (2.20) can be written in a more compact matrix form:

\[ \omega A \bar{Q} + \bar{R} = 0, \]  
(2.21)

where

\[ A = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & -2 & 0 & \cdots & 0 & -3 & \cdots \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} Q_C_1 \\ Q_C_2 \\ \vdots \\ Q_C_N \\ Q_S_1 \\ Q_S_2 \\ \vdots \\ Q_S_N \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} R_C_1 \\ R_C_2 \\ \vdots \\ R_C_N \\ R_S_1 \\ R_S_2 \\ \vdots \\ R_S_N \end{bmatrix}. \]  
(2.22)

A solution of the matrix equation Eq. (2.21) would give us Fourier coefficients \( \bar{Q} \) from the variable \( \bar{Q} \), however solving Eq. (2.21) is not desirable because \( \bar{R} \) contains differential operators that would be needed to transform into frequency domain. Since the Fourier series
expansion, Eq. (2.14) can be regarded as Discrete Fourier Transform (DFT), matrix representation of DFT is defined to switch between time and frequency domain easily:

\[ Q = E \varphi. \]  

(2.23)

\( \varphi \) represents a vector of discrete time values of \( \varphi \). Thus, with the usage of matrix representation DFT, Eq. (2.21) can be written in terms of time domain variable as:

\[ \omega A E + E R = 0. \]  

(2.24)

Since the evaluation of sources and fluxes in the frequency domain is computationally expensive, Eq. (2.24) is multiplied with \( E^{-1} \) from the left which gives us a time-spectral form of that equation:

\[ \omega E^{-1} A E + R = 0. \]  

(2.25)

Eq. (2.25) presents a set of quasi-steady state equations coupled by off-diagonal elements of the matrix \( E^{-1} A E \). If we compare Eq. (2.15) and Eq. (2.25), it can be noticed that the time derivative term is replaced by a source term. Label \( S(\varphi) \) is defined to represent HB temporal coupling vector as:

\[ S(\varphi) = \omega E^{-1} A E \varphi, \]  

(2.26)

while \( S_l \) is \( l \)-th element of the vector defined as:

\[ S_l(\varphi) = -\frac{2 \omega}{2N+1} \left( \sum_{k=1}^{2N} P_{k-1} \varphi_k \right), \text{ for } l = 1 \ldots 2N+1. \]  

(2.27)

In Eq. (2.27), \( t_k \) represents the \( k \)-th discrete time instant:

\[ t_k = \frac{kT}{2N+1}, \text{ for } k = 1, 2 \ldots 2N+1, \]  

(2.28)

where \( T \) is the base period of oscillation corresponding to \( \omega \). Furthermore, \( P \) in Eq. (2.27) denotes constant periodic coupling matrix defined as:

\[ P_r = \sum_{m=1}^{N} r \sin(rm\omega \Delta t), \quad r = -N \ldots N, \]  

(2.29)

where \( \Delta t = T/(2N+1) \).

From this brief presentation of the Harmonic Balance method, it can be seen that one transient problem is transformed into a set of \( 2N+1 \) quasi-steady state problems. Each of \( 2N+1 \) steady state problems describes one equidistantly spaced time instant of period \( T \). The steady state equations are coupled through the source term, defined by Eq. (2.27), while
convection, diffusion and source terms of the transient equation remain unchanged. In the following text benefits of the implicit coupling of the HB source term will be presented via von Neumann stability analysis. Stability analysis of implicit coupling was conducted by Gatin et al. [14].

2.4.1 Implicit coupling of the HB source term

The implicit source coupling between quasi steady-state equations is achieved through block matrix [19]. The diagonal elements of the block matrix are matrices of size \((2N+1) \times (2N+1)\), containing the diagonal coefficients of all steady-state equations, and their coupling in off-diagonal coefficients. The solution and source vectors are \((2N+1)\) vectors. It is important to say that the block matrix is used only to resolve coupling arising from HB method, while velocity, pressure and surface capturing equations are solved in a segregated manner.

We are using von Neumann stability analysis to show benefits of the implicit coupling. The von Neumann stability analysis is conducted on a one-dimensional convection problem in HB form [14]:

\[
\frac{\partial \mathcal{Q}}{\partial \tau} = \mathcal{S}(\mathcal{Q}) - u \frac{\partial \mathcal{Q}}{\partial x}, \tag{2.30}
\]

where \(\frac{\partial \mathcal{Q}}{\partial \tau}\) presents quasi-temporal term used to facilitate marching towards the steady-state solution, while \(u\) denotes velocity vector field at individual discrete time instants. In order to use the von Neumann stability analysis the source term \(\mathcal{Q}\) of HB method must be formulated in more general form. Namely HB method presented in the text above is formulated with Fourier series truncated to real part (see Eq. (2.14)), while here a complex Fourier series form is used that produces an imaginary unit in the temporal derivative. Complex Fourier series will be presented in more detail in the next chapter. With the usage of complex Fourier series Eq. (2.30) assumes the following form:

\[
\frac{\partial \mathcal{Q}}{\partial \tau} = -i \omega E^{-1} A E \mathcal{Q} - u \frac{\partial \mathcal{Q}}{\partial x}, \tag{2.31}
\]

where \(i\) stands for imaginary unit, \(i = \sqrt{-1}\). As stated earlier, each vector in Eq. (2.31) consists of values at the individual time instants, therefore Eq. (2.31) presents a set of coupled equations. However, our goal is to examine individual equation in stability analysis, hence Eq. (2.31) has to be written in a decoupled manner. Diagonalization of the equation is achieved through the following procedure: first we use expression \(\mathcal{Q} = E^{-1} Q\) and then multiply equation by \(E\) from left. Furthermore, velocity is linearised as \(u = UL + \delta u\), where \(U\) represents the mean convective velocity, \(L\) is identity vector, and \(\delta u\) is the perturbation of velocity. After this procedure Eq. (2.31) assumes the following form:

\[
\frac{\partial Q}{\partial \tau} = -i \omega A Q - U \frac{\partial Q}{\partial x}, \tag{2.32}
\]
where $Q$ represents transformed variable in frequency domain. Since matrix $A$ defined by Eq. (2.22) is a diagonal matrix, the system of equations described by Eq. (2.32) is decoupled. Hence, we can observe each of $l$ equations independently:

$$\frac{\partial Q_l}{\partial \tau} = -i\omega_l Q_l - U \frac{\partial Q_l}{\partial x}. \tag{2.33}$$

Before the analysis of implicit approach, explicit approach for coupling the HB source term is examined. In both explicit and implicit approach, implicitly treated convection term is discretised with first order upwind scheme, while quasi-temporal term is discretised with first order accurate Euler scheme. Superscript $n$ denotes the new time step value, while $o$ denotes the old time step value.

The discretised form of Eq. (2.33) for explicit coupling of HB source term is:

$$\frac{Q^n_{l,p} - Q^o_{l,p}}{\Delta \tau} = -i\omega_l Q^o_{l,p} - U \frac{Q^n_{l,p} - Q^o_{l,p-1}}{\Delta x}, \tag{2.34}$$

where $p$ stands for the $p$-th grid point i.e cell centre. The von Neumann analysis uses Fourier transform to present the distribution of a variable in space:

$$Q_{l,p} = \sum_{q=-N_c}^{N_c} V_{l,q} e^{ipq\pi/N_c}, \quad p = 1 \ldots N_c, \tag{2.35}$$

where $N_c$ stands for the total number of cells and $V_{l,q}$ is the vector of $q$-th Fourier amplitudes. The system is stable when the magnitude of the amplification factor $G = \frac{V^n_{l,q}}{V^o_{l,q}}$ is smaller than one for every $q$. An arbitrary Fourier coefficient can be used since the stability condition $G < 1$ must be valid for each Fourier coefficient:

$$Q_{l,p} = V_l e^{ip\alpha}, \tag{2.36}$$

where $\alpha$ denotes a general phase angle. Substituting Eq. (2.36) into Eq. (2.34) yields:

$$\frac{V^n_l e^{ip\alpha} - V^o_l e^{ip\alpha}}{\Delta \tau} = -i\omega_l V^o_l e^{ip\alpha} - U \frac{V^n_l e^{ip\alpha} - V^o_l e^{i(p-1)\alpha}}{\Delta x}. \tag{2.37}$$

The amplification factor is obtained after simplification of Eq. (2.37):

$$\left| \frac{V^n_l}{V^o_l} \right| = \frac{|1 - i\Delta \tau \omega_l|}{|1 + \frac{U \Delta \tau}{\Delta x} (1 - e^{-i\alpha})|}. \tag{2.38}$$

Eq. (2.38) can be written in a more suitable form after introduction of Courant-Friedrich-Lewy number $Co = U \Delta \tau / \Delta x$:

$$\left| \frac{V^n_l}{V^o_l} \right| = \frac{|1 - iCo \Delta \tau \omega_l|}{|1 + Co (1 - e^{-i\alpha})|}. \tag{2.39}$$

Eq. (2.39) tells us that solution is conditionally stable. By observing the numerator, it can
be seen that solution stability decreases with higher base frequency, coarser grid (larger $\Delta x$) and smaller convective velocity. Deterioration of stability for larger number of harmonics can limit the spectral resolution of the simulation, and convective velocity poses a limit for practical applications since the mean velocity cannot be small or zero.

The discretised form of Eq. (2.33) for implicit coupling of the HB source term is:

$$
\frac{Q^n_{l,p} - Q^o_{l,p}}{\Delta \tau} = -i \omega l Q^n_{l,p} - U \frac{Q^n_{l,p} - Q^{n-1}_{l,p}}{\Delta x},
$$

(2.40)

from which, by analogy with Eq. (2.37), Eq. (2.38) and Eq. (2.39) follows:

$$
\left| \begin{array}{c}
V^n_l \\
V^o_l
\end{array} \right| = \frac{1}{\left|1 + i \alpha \Delta \tau \omega U + \alpha \left(1 - e^{-i \alpha}\right)\right|}.
$$

(2.41)

Eq. (2.41) tells us that the solution that arises from implicit HB source term coupling is unconditionally stable. Contrary to the explicit coupling, stability of the implicit coupling solution increases with a coarser spatial grid, higher base frequency, smaller convective velocity and a larger number of harmonics. In conclusion, it can be said that implicit coupling enables stable simulations with low computational demands, and it enables simulating flows with low mean velocity. It should be noted that this analysis was conducted on linearised equation set, where convective velocity remains constant from time-step to time-step.

### 2.5 Governing equations in HB form

After the basics of HB method were presented in the previous section, in this section the governing equations of the flow (see Eq. (2.11), Eq. (2.12) and Eq. (2.13) ) will be presented in HB form. It was shown that only terms with rate-of-change are replaced with HB source terms, while other terms remained unchanged.

The continuity equation (2.11) does not have any rate-of-change terms, therefore the form of the equation remains unchanged:

$$
\nabla \cdot u_{p,l} = -\nabla \cdot u_{I,l}, \quad l = 1 \ldots 2N + 1.
$$

(2.42)

The only difference between Eq. (2.42) and Eq. (2.11) is that in Eq. (2.42) velocity field is defined in equidistantly spaced time instants, thus giving us $2N + 1$ mutually independent equations.

The momentum equation (2.12) has two time-derivative terms that need to be replaced with HB source terms:

$$
S_l(u_P) + \nabla \cdot (u_I u_P) - \nabla \cdot (v_{c_l} \nabla u_P) = \\
- S_l(u_I) - \nabla \cdot (u_I u_{I,l}) + \nabla \cdot (v_{c_l} \nabla u_{I,l}) - \frac{1}{\rho_l} \nabla p_{d,l} + \nabla u_{I,l} \cdot \nabla v_{c_l}, \quad l = 1 \ldots 2N + 1.
$$

(2.43)
Note that each of $2N + 1$ quasi-steady-state equations has corresponding density $\rho_l$ and effective kinematic viscosity $\nu_e$ fields. Since two HB source terms in Eq. (2.43) are linear operators they could be rewritten as one term but this is not done in order to allow different treatment of source terms for the perturbation velocity field $u_P$ and the incident velocity field $u_I$. Source term of the perturbation velocity field $u_P$ is treated implicitly, while source term for the incident field $u_I$ is treated explicitly.

Same as the momentum equation Level Set equation (2.13) has two time-derivative terms, therefore transformation of Level Set equation is similar to transformation of momentum equation:

$$
S_l(\psi_P) + \nabla \cdot (c_l \psi_P) - \psi_P \nabla \cdot c_l - b \nabla \cdot (\nabla \psi_P) = \\
-S_l(\psi_I) - \nabla \cdot (c_l \psi_I) + \psi_I \nabla \cdot c_l + b \nabla \cdot (\nabla \psi_I) + b \frac{\sqrt{2}}{\varepsilon} \tanh \left( \frac{\psi_l}{\varepsilon \sqrt{2}} \right) .
$$

Again, similar to Eq. (2.43), the HB source term is separated for the incident and perturbation parts of the solution to enable implicit treatment of the perturbation source coupling.

### 2.6 Closure

This chapter shows the mathematical model used to simulate two-phase, nonlinear and viscous flow in the frequency domain. With the HB method a transient problem is transformed into a set of steady-state problems. Furthermore, it is shown that implicit coupling of the HB source term yields theoretically unconditionally stable solution. In practice it is shown by Gatin et al. [14] that the solution is stable only for some minimum value of the reduced frequency. The reader is referred to [14] for details.

Since this chapter presented the governing equation for fluid flow, the next chapter will be dedicated to rigid body dynamics.
3 Rigid body dynamics

3.1 Introduction

In the field of multibody dynamics various classes of problems can be distinguished. One of these classes is forward dynamics [20] which deals with determination of the motion of mechanical system that is subjected to prescribed forces and torques. Although a mechanical system usually consists of more interconnected rigid bodies, a system in this work consists of only one rigid body. A rigid body, an ideal representation of a real physical object, is defined as body that does not deform. Therefore, the distance between any two given points on a rigid body remains constant in time regardless of external forces acting on it. A rigid body can also be viewed as a body composed of an infinite number of particles, constrained not to move with respect to each other [21]. Latter definition of a rigid body is used in a determination of its position.

Since our system consists of only one rigid body, change of its configuration can be determined by solving a system of two ordinary differential equations. The first differential equation describes translational motion, and it is called Newton equation, while second equation called Euler equation describes the rotational motion of a rigid body. Mathematically, rigid body motion is nothing more than a coordinate system transformation. Mathematical description of a rigid body motion demands introduction of two coordinate systems: inertial and body-fixed. If a certain condition is met when a body-fixed coordinate system is selected, the translation of a rigid body can be represented as translation of a body-fixed coordinate system in relation to the inertial coordinate system, while rotation of a rigid body can be represented as change of orientation of a body-fixed coordinate system. Also, when the mentioned condition is fulfilled Newton and Euler equations are decoupled.

As it was stated before the main goal of forward dynamics is to determine the motion of a mechanical system, in this case a rigid body. The motion of a rigid body is determined if its configuration, i.e. position and orientation is known at every time instant. Therefore this chapter is divided into two sections: first, governing equations for determination of the rigid body position will be discussed and then governing equations for determination of the rigid body orientation.

It is known that it is much easier to describe a translational motion than the rotational motion. Reason for that is that the translations "live" in a linear three-dimensional Euclidean space $\mathbb{R}^3$, while rotations "live" in a curved space of group $SO(3)$.

3.2 Determination of the rigid body position

During translation (change of position) of a rigid body angles between basis vectors of inertial and body-fixed coordinate system remain unchanged. Since a rigid body is defined as a sum of
an infinite number of mutually constrained particles and because angles between inertial and body-fixed system are constant, translation of a whole body can be described by translation of any point on the rigid body. The basic thesis of continuum mechanics states that one particle takes up only one point in space and that in one point in space can be only one particle. Hence, terms particle and point are used interchangeably in this work. Although any point on a body can be selected to track translation of a body, usually the centre of mass is selected. Also origin of a body-fixed reference frame must be placed in centre of mass so that Newton and Euler equation would be decoupled.

The path that body follows during translation is best described with a position vector \( \mathbf{x} \in \mathbb{R}^3 \). Vector \( \mathbf{x} \) connects origin of an inertial coordinate system to the origin of the body-fixed coordinate system i.e. centre of mass of a rigid body. As it was stated before, a general equation that governs translational motion is the Newton equation:

\[
m \ddot{\mathbf{x}} = \mathbf{f}_R,
\]

where \( m \) denotes body mass, \( \ddot{\mathbf{x}} \) is acceleration of centre of mass, while \( \mathbf{f}_R \) stands for resultant force acting in the centre of mass of the body.

If we want to model a linear spring-damper system acting on the body, Eq. (3.1) must be modified. From the resultant force \( \mathbf{f}_R \), force in the spring is extracted:

\[
\mathbf{f}_s = \mathbf{c} \mathbf{x},
\]

as well as the force of viscous damping:

\[
\mathbf{f}_d = \mathbf{k} \dot{\mathbf{x}}.
\]

In Eq. (3.2), \( \mathbf{c} \) stands for diagonal stiffness matrix of the following form:

\[
\mathbf{c} = \begin{bmatrix}
c_x & 0 & 0 \\
0 & c_y & 0 \\
0 & 0 & c_z
\end{bmatrix},
\]

where \( c_x, c_y \) and \( c_z \) are stiffness coefficients of linear spring in the direction of \( x, y \) and \( z \) axis respectively, while \( \mathbf{x} \) is a position vector. Analogous in Eq. (3.3), \( \mathbf{k} \) stands for diagonal damping matrix of the following form:

\[
\mathbf{k} = \begin{bmatrix}
k_x & 0 & 0 \\
0 & k_y & 0 \\
0 & 0 & k_z
\end{bmatrix},
\]

where \( k_x, k_y \) and \( k_z \) are damping coefficients of viscous damper in the direction of \( x, y \) and \( z \) axis respectively, while \( \dot{\mathbf{x}} \) is a relative velocity vector.
Note that equations (3.2) and (3.3) are giving us only the magnitude of linear spring and viscous damper force, while the direction is always opposite to the direction of position and relative velocity vectors, respectively. If we take into account direction of spring and damper forces, Eq. (3.1) take the following form:

\[ m\ddot{x} + k\dot{x} + \xi x = f, \]  

where \( m \) stand for diagonal mass matrix:

\[
   m = \begin{bmatrix}
      m & 0 & 0 \\
      0 & m & 0 \\
      0 & 0 & m 
   \end{bmatrix}.
\]  

(3.7)

Notation of mass in a form of the mass matrix is used to be consistent with notations of stiffness and damping matrix.

### 3.2.1 Frequency domain

In order to solve translational motion in frequency domain, all time-domain variables from Eq. (3.6) must be expanded into Fourier series. Same as in the von Neumann stability analysis, complete complex Fourier series is used. General variable \( \mathcal{Q}(t) \) expanded into the Fourier series with \( N \) harmonics takes the following form:

\[
   \mathcal{Q}(t) = Q_0 + \sum_{p=1}^{N} Q_p e^{ip\omega t},
\]

(3.8)

where \( i \) stands for imaginary unit, \( i = \sqrt{-1} \), \( Q_p \) denotes complex Fourier amplitude of \( p \)-th harmonic, while \( \omega \) denotes base angular frequency. Since Eq. (3.6) has terms with both first and second derivative it is necessary to differentiate Fourier series of general variable (Eq. (3.8)). The first derivative of Eq. (3.8) takes the following form:

\[
   \dot{\mathcal{Q}}(t) = \sum_{p=1}^{N} ip\omega Q_p e^{ip\omega t},
\]

(3.9)

while the second derivative takes form:

\[
   \ddot{\mathcal{Q}}(t) = \sum_{p=1}^{N} -p^2 \omega^2 Q_p e^{ip\omega t}.
\]

(3.10)

Insertion of the Fourier series of the position vector \( \mathbf{x} \) and his derivatives into Eq. (3.6) yields governing equation for translation motion in the frequency domain:
\[
\sum_{p=1}^{N} \left( -p^2 \omega^2 X_p e^{ip\omega t} + \frac{k}{2} \sum_{p=1}^{N} ip \omega X_p e^{ip\omega t} + c \sum_{p=1}^{N} X_p e^{ip\omega t} \right)
+ c X_0 = F_0 + \sum_{p=1}^{N} F_p e^{ip\omega t},
\]

where \( X_p \) stands for complex Fourier coefficients of position vector \( x \). For each of \( N \) harmonics exists one vector \( X_p \) that contains Fourier coefficients in the direction of \( x, y \) and \( z \) axis. After equating the terms with the corresponding frequencies in Eq. (3.11), following equation is obtained:

\[
X_{pj} = \frac{F_{pj}}{-mp^2 \omega^2 + ip \omega k_j + c_j}, \quad p = 1 \ldots N \text{ and } j = x, y, z.
\]

Eq. (3.12) is used to calculate the magnitude of position Fourier coefficients with the respect to the inertial coordinate system, i.e. with respect to the direction of \( x, y \) and \( z \) axis of inertial coordinate system.

The velocity is defined as the rate-of-change of position vector:

\[
u = \frac{dx}{dt} = \dot{x}.
\]

If Eq. (3.8) and Eq. (3.9) are used to transform time domain variables \( u \) and \( x \) into frequency domain, Eq. (3.13) takes the following form:

\[
U_0 + \sum_{p=1}^{N} U_p e^{ip\omega t} = \sum_{p=1}^{N} ip \omega X_p e^{ip\omega t}.
\]

After equating the terms with the same harmonic in Eq. (3.14), equation used for calculation of the velocity Fourier coefficients from position Fourier coefficient is obtained:

\[
U_p = ip \omega X_p, \quad p = 1 \ldots N,
\]

where \( U_p \) stands for complex Fourier coefficients of velocity vector \( u \). Similar to the Fourier coefficient position vector, Fourier coefficient velocity vector contains Fourier coefficients of velocity with respect to the inertial coordinate system.

### 3.3 Determination of a rigid body orientation

Orientation of a body is always defined with respect to the inertial coordinate system, while the rotation (change of orientation) is just change of angles between basis vectors of inertial and body-fixed coordinate systems. In the previous section it is stated that in order to have decoupled Newton and Euler equations, origin of the body-fixed coordinate system must be
in the centre of mass of the rigid body. Also, in order to have diagonal tensor of inertia, a
direction of a body-fixed coordinate system must be in the direction of the principal axes of
inertia of a rigid body. Principal axes of inertia can be found for every rigid body, and they can
be easily found in symmetrical bodies because every axis perpendicular to the symmetry plane
is the principal axis. If a body is three-times symmetrical all principal axes are defined, but for
the two-times and one-time symmetrical bodies only two principal axes or one principal axis is
defined beforehand, respectively, while the others must be determined through computation.

The main difference between translations and rotations is the space in which they are
defined in. As it was stated before translations are described with the position vector $\mathbf{x}$ which
is the part of Euclidean space. Euclidean space is linear in its nature. Rotations are
unamibiguously defined with the rotation matrix $R$ which is the part of special orthogonal
group $SO(3)$. $SO(3)$ group is curved in its nature.

A rotation matrix $R$ in 3D space is given by a $3 \times 3$ matrix. Hence, $R$ has 9 elements.
However, these elements are not all independent. Therefore, different parametrisations of the
rotation matrix are used in dynamics. One of the parametrisation is Euler angles. The problem
with Euler angles is that they are a parametrisation of rotation matrix in a certain domain; thus
they cannot describe all rotations. This problem is known as the gimbal lock. Another widely
used parametrisation is quaternion-based parametrisation. Quaternions are a single example of
a more general class of hypercomplex numbers discovered by Hamilton [22]. Quaternions
parametrisate rotation matrix with 4 parameters, and they parametrisate the whole domain of
rotation matrix. Hence, they are more suitable for use than Euler angles. However, trend in
multibody dynamics in recent years is to utilize the fact that rigid body motions form a Lie
group. In the following text only brief presentation about derivation of rotation matrix from
Lie group approach is given. More detailed presentation exceeds the extends of this thesis.
Therefore, reader can find detailed derivation of Geometric methods and formulations in
computational multibody system dynamics in the paper by Müller et al. [15].

Configuration of a rigid body is represented by a rotation matrix $R \in SO(3)$ and a position
vector $\mathbf{x} \in \mathbb{R}^3$ and it can be denoted by $C = (R, \mathbf{x}) \in SE(3)$. The Lie group $SE(3)$ is a semidirect
product of the rotation group $SO(3)$ and the translation group $\mathbb{R}^3$:

$$SE(3) = \{ C = (R, \mathbf{x}) \mid R \in SO(3), \mathbf{x} \in \mathbb{R}^3 \}.$$  \hfill (3.16)

Rotation group $SO(3)$ acts on the translation group $\mathbb{R}^3$ and that describes coordinate system
transformations. The rotation group, i.e. special orthogonal group $SO(3)$ that unambiguously
describes rotations, is Lie group [21] defined as:

$$SO(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid R^{-1} = R^T, \det R = 1 \}.$$  \hfill (3.17)

The frame transformations are often represented in the form of homogenous transformation
matrices:

\[
\mathbf{C} = \begin{bmatrix}
\mathbf{R} & \mathbf{x} \\
0 & 1
\end{bmatrix} \in \text{SE}(3),
\]  

where \( \text{SE}(3) \) is matrix Lie group.

In the paper by Müller et al. [15] it is shown that configuration \( \mathbf{C} \) is equal to the closed form of the exponential mapping on \( \text{SE}(3) \) which can be found by evaluating the matrix exponential:

\[
\exp\left(\hat{\mathbf{X}}\right) = \begin{bmatrix}
\exp\left(\hat{\xi}\right) & \text{dexp}\hat{\eta} \\
0 & 1
\end{bmatrix},
\]  

(3.19)

with Euler-Rodrigues formula:

\[
\exp\left(\hat{\xi}\right) = \mathbf{I} + \frac{\sin \| \hat{\xi} \|}{\| \hat{\xi} \|} \hat{\xi} + \frac{1 - \cos \| \hat{\xi} \|}{\| \hat{\xi} \|^2} \hat{\xi}^2,
\]  

(3.20)

and the right-trivialized differential of the exponential mapping on \( \text{SO}(3) \), i.e. \( \text{dexp}\hat{\xi} \).

Since the configuration \( \mathbf{C} \) Eq. (3.18) is equal to the exponential mapping \( \exp\left(\hat{\mathbf{X}}\right) \), Eq. (3.19), it can be concluded by comparing these two equations that the rotation matrix can be evaluated as:

\[
\mathbf{R} = \mathbf{I} + \frac{\sin \| \hat{\xi} \|}{\| \hat{\xi} \|} \hat{\xi} + \frac{1 - \cos \| \hat{\xi} \|}{\| \hat{\xi} \|^2} \hat{\xi}^2.
\]  

(3.21)

In Eq. (3.21) \( \mathbf{I} \) stands for identity matrix of size 3 \( \times \) 3, while \( \hat{\xi} \) is a variable calculated from angular velocity \( \hat{\omega} \):

\[
\hat{\xi} = \hat{\omega}.
\]  

(3.22)

Eq. (3.22) ensures second order of accuracy in the numerical procedure. The explanation about other variables in Eq. (3.19) can be found in [15].

Angular velocity \( \hat{\omega} \) used in Eq. (3.22) is evaluated from the Euler equation. The Euler equation is the governing equation for the rotation of a rigid body:

\[
\mathbf{J} \dot{\hat{\omega}} + \hat{\omega} \times \mathbf{J} \hat{\omega} = \mathbf{l},
\]  

(3.23)

where \( \mathbf{J} \) denotes tensor of inertia, while \( \hat{\omega} \) stands for angular velocity and \( \mathbf{l} \) stands for resultant torque acting on a body. Note that \( \hat{\omega} \) used in the Euler equation is different from \( \hat{\omega} \) used in Fourier series. Former is vector quantity that stands for angular velocity, while latter is a scalar quantity that denotes the base frequency of oscillation. It was stated before that with the correct selection of a body-fixed coordinate system, the tensor of inertia becomes diagonal:
where $J_{xx}$, $J_{yy}$ and $J_{zz}$ are principal moments of inerta.

Cross product in the Euler equation (Eq. (3.23)) is not suitable for numerical use. Hence, the cross product should be written as matrix multiplication. Cross product is written as matrix multiplication with the skew-symmetrical matrix:

$$
\tilde{\omega} = \begin{bmatrix}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & -\omega_x \\
-\omega_y & \omega_x & 0 
\end{bmatrix}.
$$

(3.25)

After the cross product is written as a matrix multiplication Euler equation takes the following form:

$$
\dot{J} \omega + \tilde{\omega} J \omega = l.
$$

(3.26)

Although, the rotation matrix $R$ is used to rotate a rigid body, we need a more intuitive way to express orientation of a body in space. For that purpose we use Euler angles. It has been said that the use of Euler angles is not suitable for describing rotations, but that is related to the use of Euler angles for the formation of rotation matrix $R$. Angular velocity $\omega$ can be represented through derivation of angle $\theta$, but because rotations are part of curved space we can not simply say that angular velocity is equal to the derivation of angle i.e. $\omega \neq \dot{\theta}$. If we insert this representation of angular velocity via derivation of angles in the Euler equation (Eq. (3.26)), its solution will give us angles and not angular velocity. Those angles are then used to form rotation matrix $R$. Rotation matrix can take several different forms based on selected sequence of rotations. Here we use following sequence of rotations: $\theta_x$ about $x$-axis, then $\theta_y$ about the $y$-axis, then $\theta_z$ about the $z$-axis, each rotation being applied about one of the axes of global coordinate system as opposed to one of axes of the body-fixed coordinate system [23]:

$$
R = \begin{bmatrix}
\cos \theta_x \cos \theta_z & \cos \theta_y \sin \theta_z & -\sin \theta_x \\
\sin \theta_x \sin \theta_y \cos \theta_z - \cos \theta_x \sin \theta_z & \sin \theta_x \sin \theta_y \sin \theta_z + \cos \theta_x \cos \theta_z & \sin \theta_x \cos \theta_y \\
\cos \theta_x \sin \theta_y \sin \theta_z + \sin \theta_x \cos \theta_z & \cos \theta_x \sin \theta_y \cos \theta_z - \sin \theta_x \sin \theta_z & \cos \theta_x \cos \theta_y 
\end{bmatrix}.
$$

(3.27)

Since in this work the rotation matrix is calculated with Eq. (3.21), Eq. (3.27) is used to determinate Euler angles $\theta_x$, $\theta_y$ and $\theta_z$. Hence Euler angles are equal to:
\[ \theta_x = \arctan \frac{R_{yz}}{R_{zz}}, \]
\[ \theta_y = \arctan \frac{-R_{xz}}{c_2}, \]
\[ \theta_z = \arctan \frac{\sin \theta_x R_{yz}}{\cos \theta_x R_{yy} - \sin \theta_x R_{zy}}, \]

where \( c_2 = \sqrt{R_{xx}^2 + R_{xy}^2} \).

### 3.3.1 Frequency domain

The goal of transforming Euler equation to frequency domain is to get the expression for Fourier coefficients of angular velocity. The Euler equation (3.26) will be written in the more suitable form before Fourier series expansion is used. First, the second term on the left-hand side of Eq. (3.26) is transferred to the right-hand side, and then the whole equation is multiplied with \( J^{-1} \):

\[ \dot{\omega} = J^{-1} \left[ l + (J\omega) \cdot \omega \right], \]

(3.29)

Note that the vector that forms the skew-symmetrical matrix in transferred term is changed. After all time domain variables are transformed in the frequency domain via Fourier series expansion (Eq. (3.8) and Eq. (3.9)), the Euler equation takes the following form:

\[ \sum_{p=1}^{N} \frac{ip \omega \Omega_p e^{ip\omega t}}{J^{-1}} = L_0 + \sum_{p=1}^{N} L_p e^{ip\omega t} + * \left[ J^{-1} \left( \Omega_0 + \sum_{p=1}^{N} \Omega_p e^{ip\omega t} \right) \right] \left( \Omega_0 + \sum_{p=1}^{N} \Omega_p e^{ip\omega t} \right), \]

(3.30)

For clarity, skew-symmetrical matrix in Eq. (3.30) is denoted with asterisk * rather than with tilde \( \sim \). Equalizing the terms with the same harmonic and additional simplification yields a piecewise expression for the Fourier coefficients of angular velocity:

\[ \Omega_{px} = \frac{1}{ip \omega J_{xx}} \left[ L_{px} + (J_{yx} - J_{zz}) \sum_{k=0}^{p} \Omega_{p} \Omega_{(p-k)z} \right], \]

\[ \Omega_{py} = \frac{1}{ip \omega J_{yy}} \left[ L_{py} + (J_{zz} - J_{xx}) \sum_{k=0}^{p} \Omega_{p} \Omega_{(p-k)x} \right], \quad p = 1 \ldots N, \]

(3.31)
\[
\Omega_{pz} = \frac{1}{ip\omega J_{zz}} \left[ L_{pz} + (J_{xx} - J_{yy}) \sum_{k=0}^{p} \Omega_{px} \Omega_{(p-k)y} \right],
\]
where \( \Omega_{px}, \Omega_{py}, \text{and} \Omega_{pz} \) stand for Fourier coefficient of angular velocity in the direction of \( x, y \) and \( z \) axis respectively, while analogous to the angular velocity \( L_{px}, L_{py}, \text{and} \ L_{pz} \) are Fourier coefficients of torque in the direction of \( x, y \) and \( z \) axis.

The rotation matrix is evaluated in time domain (3.21), but the time domain variable \( \xi \) is obtained from its frequency domain counterpart via inverse Fourier transform. The Fourier coefficients of the variable \( \xi \) are calculated from Eq. (3.22) transformed into frequency domain:

\[
\sum_{p=1}^{N} ip\omega \Xi_p e^{ip\omega t} = \Omega_0 + \sum_{p=1}^{N} \Omega_p e^{ip\omega t},
\]
where \( \Xi_p \) stands for Fourier coefficient of variable \( \xi \).

Equalizing the terms of the corresponding harmonics yields the expression for Fourier coefficients of variable \( \xi \):

\[
\Xi_{pj} = \frac{\Omega_{pj}}{ip\omega}, \quad \text{for} \quad p = 1 \ldots N \quad \text{and} \quad j = x, y, z.
\]

### 3.4 Forces acting on a rigid body

The rigid body changes its configuration due to the forces and torques that act on it. Acting force on a body submerged in a fluid can be divided into two parts: part of the force due to pressure \( f_p \) and part of the force due to viscous effects \( f_v \):

\[
f = f_p + f_v. \tag{3.34}
\]

Analogous to the force, torque can also be divided into pressure and viscous part:

\[
l = l_p + l_v. \tag{3.35}
\]

Forces and torques in the time-domain are calculated in the global coordinate system as:

\[
f_p = \sum_{b_f} s_{bf} f_{pf} + mg,
\]

\[
f_v = \sum_{b_f} p_{bf} v_{e,f} s_{bf} T_s^*,
\]

\[
l_p = \sum_{b_f} r_{bf} \times s_{bf} f_{pf},
\]
\[ \mathbf{f}_v = \sum_{b_f} r_f \times \left( \rho_f \nu_{e,f} \mathbf{f}_e \cdot \mathbf{T}_e^* \right), \quad (3.39) \]

where \( \sum_{b_f} \) denotes summation over all body faces, \( \rho_f \) is the corresponding density at the boundary face and \( \nu_{e,f} \) is effective kinematic viscosity. \( T^*_e \) is deviatoric part of stress tensor \( T_e \), which is defined as twice symmetric part of the \( \nabla \mathbf{u} \) tensor. Vector \( r_f \) is the distance vector that connects current boundary face and centre of mass of a body.

Since the fluid flow is solved with HB method all flow fields are calculated at \( 2N + 1 \) equidistantly spaced time instants of period \( T \). Hence, forces and torques, which are calculated with pressure \( p \) and velocity field \( \mathbf{u} \), are known only in these discrete time instants. Therefore, forces and torques calculated in time domain must be transformed into frequency domain via DFT:

\[
\mathcal{Q}_0 = \frac{1}{2N+1} \sum_{j=0}^{2N} \mathcal{Q}_j, \\
\mathcal{Q}_p = \frac{2}{2N+1} \sum_{j=0}^{2N} \mathcal{Q}_j e^{i \frac{2\pi pj}{2N+1}}, \quad p = 1 \ldots N, \quad (3.40)
\]

where \( \mathcal{Q}_p \) stands for Fourier coefficient of \( p \)-th harmonic, while \( \mathcal{Q}_j \) represents time variable at \( j \)-th time instant. After the transformation force and torque Fourier coefficients can be used to calculate the motion of a rigid body.

For the sake of complete presentation, an expression for inverse DFT used in the next chapter is also given here:

\[
\mathcal{Q}_j = \sum_{p=0}^{N} \mathcal{Q}_p e^{-i \frac{2\pi pj}{2N+1}}, \quad j = 1 \ldots 2N + 1. \quad (3.41)
\]

### 3.5 Closure

This section covered rigid body dynamics. The main goal was to represent the method for calculation of body motion that is compatible with the Harmonic Balance method. For this purpose governing equations for translational and rotational motion, i.e. the Newton and Euler equation, are transformed in the frequency domain. Furthermore, detailed derivation of expressions which are used to obtain position and velocity in spectral space is given.
4 Validation of spectral rigid body motion

4.1 Introduction

Method for calculating rigid body motion presented in the previous chapter will be tested on two test cases. In this chapter the validation of the rigid body motion in frequency domain will be conducted, while in the next chapter results will be compared to a transient simulation. All simulations are performed in the CFD software foam-extend.

Mathematical model presented in the previous chapter (Chapter 3), which is used to calculate body motion in frequency domain is verified against analytical results. Goal of simulations in this case is to verify the accuracy of kinematic variables, i.e. position, velocity, angle and angular velocity acquired through equations (3.12), (3.15), (3.29) and (3.28). Since only kinematic variables are tested, forces and torques acting on the body are imposed with a constant first order amplitude in the frequency domain. From Eq. (3.12) it can be seen that calculation of position Fourier coefficient of \( p \)-th harmonic depends only on value of force Fourier coefficient of the same harmonic, while it does not depend on other harmonics. Hence, value for each of the \( N \) force Fourier coefficient is set independently. Analogously, if we take a look at Eq. (3.29) it can be seen that calculation of angular velocity Fourier coefficient of \( p \)-th harmonic depends only on the value of torque Fourier coefficient of the same harmonic and therefore value for each of the \( N \) torque Fourier coefficient is also set independently.

For the numerical simulations forces and torques must be set in frequency domain, however since analytical solution is obtained from Eq. (3.6) and Eq. (3.26) in time-domain, their corresponding values in time domain also must be known. Of course, if the variable is known in one of the domains, it is easily projected into the other via Fourier transform or inverse Fourier transform. Because Fourier coefficient is an imaginary number, it consists of the real and imaginary part, while periodic function in time domain is a sum of sine and cosine terms. The real part of Fourier coefficient is the amplitude of cosine function of corresponding harmonic, while imaginary part of Fourier coefficient is the amplitude of sine function of corresponding harmonic. In order to simplify the analytical calculation, rigid body is placed in single-phase fluid flow. Furthermore, kinematic viscosity of air, which is fluid used in this case, is set to zero, therefore, damping coefficients can be neglected. This means that we only need to know material properties of the body i.e. mass \( m \) and tensor of inertia \( J \).

Validation of translational and rotational motion is conducted separately. In case of translational motion, force that acts on a body causes translation along the \( y \) axis of global coordinate system, while in case of rotational motion, torque that acts on a body causes rotation about \( z \) axis of the global coordinate system. Since this is a 2D simulation, translation along \( y \) axis is heave motion, while rotation about the \( z \) axis is pitch rotation.

Heave motion is validated with the 9 different forces. The forces differ in the number of used harmonics and in the magnitude of corresponding amplitude. Pitch rotation is also
validated with 9 torques with different number of used harmonics and different magnitude of corresponding amplitude.

4.2 Translation

The translation is tested on a rectangular rigid body placed in center of the computational domain as it can be see in Figure 4.1. Since only heave motion is allowed, periodic force is acting in the direction of $y$ axis.

![Figure 4.1: Computational domain for the validation of translational motion.](image)

*Analytical solution.* Governing equation for translational motion is the vectorial equation (3.6). Since in this case only motion in the direction of $y$ axis is allowed this equation reduces to a scalar equation. Furthermore, contribution of stiffness and damping coefficient can be neglected. Hence, equation that will be solved to obtain analytical solution is:

$$m \cdot \ddot{y}(t) = f(t),$$  \hspace{1cm} (4.1)  

where $\ddot{y}$ is the acceleration of a body in $y$ direction. All simulations will be conducted with $N = 4$ harmonics. Therefore, force can be written in general form as:

$$f(t) = \hat{f}_1^s \sin(\omega t) + \hat{f}_1^c \cos(\omega t) + \hat{f}_2^s \sin(2\omega t) + \hat{f}_2^c \cos(2\omega t) + \hat{f}_3^s \sin(3\omega t) + \hat{f}_3^c \cos(3\omega t) + \hat{f}_4^s \sin(4\omega t) + \hat{f}_4^c \cos(4\omega t)$$ \hspace{1cm} (4.2)  

$$= \sum_{n=1}^{4} (\hat{f}_n^s \sin(n\omega t) + \hat{f}_n^c \cos(n\omega t)),$$
where \( \hat{f}_n^s \) represents the \( n \)-th harmonic amplitude of sine part of the force, while \( \hat{f}_n^c \) represents the \( n \)-th harmonic amplitude of cosine part of the force. After insertion of general force (Eq. (4.2)) in the governing equation (Eq. (4.1)) and then a division by mass of a body, equation for body acceleration takes the following form:

\[
\ddot{y}(t) = \frac{1}{m} \sum_{n=1}^{4} \left( \hat{f}_n^s \sin(n \omega t) + \hat{f}_n^c \cos(n \omega t) \right)
\]

(4.3)

Equation for body velocity is obtained after integrating the acceleration equation, Eq. (4.3):

\[
\dot{y}(t) = \frac{1}{m n \omega} \sum_{n=1}^{4} \left( \hat{f}_n^s \cos(n \omega t) + \hat{f}_n^c \sin(n \omega t) \right) + C_1.
\]

(4.4)

Analogous equation for the position of the centre of mass is obtained after integrating the velocity equation, Eq. (4.4):

\[
y(t) = \frac{1}{m (n \omega)^2} \sum_{n=1}^{4} \left( -\hat{f}_n^s \sin(n \omega t) - \hat{f}_n^c \cos(n \omega t) \right) + C_1 \cdot t + C_2.
\]

(4.5)

Integration constants \( C_1 \) and \( C_2 \) are usually determined from initial conditions i.e. position and velocity of a body in initial moment, but in this case initial values are calculated with the use of basic assumptions of Harmonic Balance. The basic assumption of the Harmonic Balance is that all variables and functions are periodic. In order to validate the results of spectral motion algorithm position and velocity given with Eq. (4.4) and Eq. (4.5) will be transformed in the frequency domain. Therefore, position and velocity functions must remain periodic. Also it was stated earlier that in the existing model mean value of position and velocity in the frequency domain is not solved, i.e. it remains zero throughout the simulation. Hence, integration constant \( C_1 \) is obtained from condition of zero mean value of velocity in the time domain.

Mean value theorem for integrals reads:

\[
\bar{f}(x) = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx.
\]

(4.6)

If we use the mean value theorem on velocity equation (Eq. (4.4)) and if we take into account the fact that mean value must be zero, we get:

\[
\frac{1}{T} \int_{0}^{T} \left[ \frac{1}{m n \omega} \sum_{n=1}^{4} \left( -\hat{f}_n^s \cos(n \omega t) + \hat{f}_n^c \sin(n \omega t) \right) + C_1 \right] \, dt = 0.
\]

(4.7)

After integration and further simplification of Eq. (4.7), constant \( C_1 \) is obtained and it is equal to zero, \( C_1 = 0 \). This result is expected, since the term containing \( C_1 \) is responsible for nonperiodic linear change. Hence, in order for position to remain periodic \( C_1 \) must be zero.

Value of integration constant \( C_2 \) is obtained after applying the mean value theorem (see
Eq. (4.6)) on Eq. (4.5). Knowing that \( C_1 = 0 \), we can write:

\[
\frac{1}{T} \int_0^T \left[ \frac{1}{m} \frac{1}{(n\omega)^2} \sum_{n=1}^{4} (-\hat{f}_n^s \sin(n\omega t) - \hat{f}_n^c \cos(n\omega t)) + C_2 \right] \, dt = 0. \tag{4.8}
\]

Analogous to Eq. (4.7), after integration and further simplification of Eq. (4.8), integration constant \( C_2 \) is obtained, \( C_2 = 0 \).

Now that the integration constants are obtained we can write the final form of governing equations for heave motion:

\[
\ddot{y}(t) = \frac{1}{m} \sum_{n=1}^{4} \left( \hat{f}_n^s \sin(n\omega t) + \hat{f}_n^c \cos(n\omega t) \right),
\]

\[
\dot{y}(t) = \frac{1}{m n\omega} \sum_{n=1}^{4} \left( -\hat{f}_n^s \cos(n\omega t) + \hat{f}_n^c \sin(n\omega t) \right), \tag{4.9}
\]

\[
y(t) = \frac{1}{m (n\omega)^2} \sum_{n=1}^{4} \left( -\hat{f}_n^s \sin(n\omega t) - \hat{f}_n^c \cos(n\omega t) \right).
\]

### 4.3 Rotation

The rotation is tested on a rectangular rigid body placed in a center of a computational domain as it can be see in Figure 4.2. Since only pitch motion is allowed, periodic torque is acting in the direction of \( z \) axis.

![Figure 4.2: Validation of pitch rotation](image)

**Analytical solution.** Governing equation for the rotational motion of a rigid body is Euler
equation (3.23). In the case of planar rigid body dynamics, Euler equation can be simplified. Since the only rotation that exist is the rotation about $z$ axis, tensor of inertia $J$ can be replaced with moment of inertia $J_{zz}$. Since $J$ is now replaced with scalar value, the second term of Euler equation (3.23) consists of a cross product of colinear vectors which is equal to zero. Furthermore, in the planar dynamics angular velocity is equal to the angle differentiation:

$$\omega = \frac{d\theta}{dt}.$$  \hspace{1cm} (4.10)

Hence, Euler equation for planar dynamics takes the following form:

$$J_{zz} \ddot{\theta} = l(t),$$  \hspace{1cm} (4.11)

where $\ddot{\theta}$ stands for angular acceleration of a rigid body, while $l(t)$ stands for acting torque. Analogous to the validation of the translation, rotation is also validated with the simulations with $N = 4$ harmonics. Therefore, torque can be written in the same general form as force (Eq. (4.2)):

$$l(t) = \sum_{n=1}^{4} \left( \hat{l}_s^n \sin(n\omega t) + \hat{l}_c^n \cos(n\omega t) \right),$$  \hspace{1cm} (4.12)

where $\hat{l}_s^n$ stands for $n$-th harmonic amplitude of the sine part of torque, while $\hat{l}_c^n$ stands for $n$-th harmonic amplitude of the cosine part of torque. Equation for angular acceleration is obtained after the insertion of generalized torque and division by $J_{zz}$:

$$\ddot{\theta} = \frac{1}{J_{zz}} \sum_{n=1}^{4} \left( \hat{l}_s^n \sin(n\omega t) + \hat{l}_c^n \cos(n\omega t) \right).$$  \hspace{1cm} (4.13)

Equation for angular velocity is obtained after the integration of angular acceleration equation, Eq. (4.13):

$$\dot{\theta}(t) = \frac{1}{J_{zz}} \frac{1}{n\omega} \sum_{n=1}^{4} \left( -\hat{l}_s^n \cos(n\omega t) + \hat{l}_c^n \sin(n\omega t) \right) + C_3,$$  \hspace{1cm} (4.14)

while equation for the determination of orientation (angle) is obtained after integration of angular velocity equation, Eq. (4.14):

$$\theta(t) = \frac{1}{J_{zz} (n\omega)^2} \sum_{n=1}^{4} \left( -\hat{l}_s^n \sin(n\omega t) - \hat{l}_c^n \cos(n\omega t) \right) + C_3 \cdot t + C_4.$$  \hspace{1cm} (4.15)

Values of integration constants $C_3$ and $C_4$ are determined analogous to the determination of the values of integration constant $C_1$ and $C_2$. After the use of the mean value theorem on the Eq. (4.14) and Eq. (4.15), respectively, we get that both constants must be equal to the zero. Hence, equations used to determine analytical solution for pitch rotation are:
\[ \theta = \frac{1}{J_{zz}} \sum_{n=1}^{4} (\hat{l}_n^s \sin(n\omega t) + \hat{l}_n^c \cos(n\omega t)), \]
\[ \dot{\theta}(t) = \frac{1}{J_{zz} n\omega} \sum_{n=1}^{4} (-\hat{l}_n^s \cos(n\omega t) + \hat{l}_n^c \sin(n\omega t)), \] (4.16)
\[ \theta(t) = \frac{1}{J_{zz} (n\omega)^2} \sum_{n=1}^{4} (-\hat{l}_n^s \sin(n\omega t) - \hat{l}_n^c \cos(n\omega t)). \]

4.4 Set up of the numerical simulation

The computational domain is discretized with blockMesh utility which is a part of foam-extend with dimensions shown in Figure 4.1 (and in Figure 4.2). The finite volume mesh is shown on Figure 4.3. Mesh is made of 8 blocks, with total of 1898 hexahedra cells.

![Figure 4.3: First validation test case mesh](image)

The rigid body is placed in a single phase fluid flow. Since the flow field is not important for the force/torque calculation in this case, boundary conditions are not discussed here. The mass of the rigid body is \( m = 10 \text{kg} \), while moment of inertia is set to \( J_{zz} = 100 \text{kg} \cdot \text{m}^3 \). The period of oscillation is set to \( T = 0.7 \text{s} \), yielding the frequency of oscillation \( \omega = 8.976 \text{rad/s} \).

In the HB method every variable and field is calculated in \( 2N + 1 \) equally distanced time instants. Spacing between time instants is \( \Delta t = T / (2N + 1) \). Since in our simulations we use \( N = 4 \) harmonics and base period of oscillation is \( T = 0.7 \text{s} \), time instants in which the motion is calculated are separated by \( \Delta t = 0.0778 \text{s} \).

Since this solver is designed to simulate wave-induced ship motion, calculation of the flow is its integral part and can not be avoided, but as it was stated earlier this case is designed in a way that decouples flow and motion. In all following simulations motion starts after one hundred iterations, while the first one hundred iterations are used to ensure stable flow solution.
This is noticeable in figures showing the convergence of the numerical solution.

4.5 Results

As stated before a total of eighteen simulations are performed in order to check the accuracy of motion solution algorithm. First nine simulations are performed to check the translation, while the remaining simulations are performed to check rotation of motion. The complexity of the force and torque is increasing with the number of simulations, i.e., most complex force is in simulation No. 9, while the most complex torque is in simulation No. 18.

4.5.1 Translation

Forces used in validation of translational motion solution are presented in Table 4.1.

<table>
<thead>
<tr>
<th>Order</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation 1</td>
<td>$Re(F_{p_1})$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Im(F_{p_1})$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Simulation 2</td>
<td>$Re(F_{p_2})$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Im(F_{p_2})$</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
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</tr>
<tr>
<td></td>
<td>$Im(F_{p_3})$</td>
<td>40</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Simulation 4</td>
<td>$Re(F_{p_4})$</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Im(F_{p_4})$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>Simulation 5</td>
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<td>0</td>
</tr>
<tr>
<td></td>
<td>$Im(F_{p_5})$</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Im(F_{p_6})$</td>
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<td>15</td>
<td>0</td>
</tr>
<tr>
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<td>20</td>
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</tr>
<tr>
<td></td>
<td>$Im(F_{p_7})$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Simulation 8</td>
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<td>150</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Im(F_{p_8})$</td>
<td>0</td>
<td>-120</td>
<td>0</td>
</tr>
<tr>
<td>Simulation 9</td>
<td>$Re(F_{p_9})$</td>
<td>0</td>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>$Im(F_{p_9})$</td>
<td>50</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>Order, $p$</td>
<td>$</td>
<td>X_{p_{y}}</td>
<td>$, m</td>
<td>$</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------</td>
<td>------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation 1</td>
<td>numerical 0.0012411</td>
<td>analytical 0.0012411</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation 2</td>
<td>numerical 0.0012411</td>
<td>analytical 0.0012411</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation 3</td>
<td>numerical 0.049647</td>
<td>analytical 0.049647</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation 4</td>
<td>numerical 0.049647</td>
<td>analytical 0.049647</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation 5</td>
<td>numerical 0.044751</td>
<td>analytical 0.044751</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation 6</td>
<td>numerical 0.024823 0.22281</td>
<td>analytical 0.024823 0.22281</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation 7</td>
<td>numerical 0.012411 0.11140</td>
<td>analytical 0.012411 0.11140</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation 8</td>
<td>numerical 0.18617 1.67112</td>
<td>analytical 0.18617 1.67112</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation 9</td>
<td>numerical 0.062059 0.00930888 0.0020686 0.0034908</td>
<td>analytical 0.062059 0.00930888 0.0020686 0.0034908</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From the results in Table 4.2 it can be seen that numerical solution of translational motion converges towards analytical solution. Convergence of the Fourier coefficients of position $X_{p_{y}}$ and velocity $U_{p_{y}}$ is shown in the figures in appendix A.1, while the comparison of the numerical and analytical solution for position $y$ and velocity $u_{y}$ in time domain is given in figures in appendix A.2.
4.5.2 Rotation

Torques used in validation of rotational motion solution are presented in Table 4.1. To avoid confusion with the translational simulations, rotational simulations are denoted with the numbers ranging from 10 to 18.

Table 4.3: Torques in the simulations for rotational motion validation

<table>
<thead>
<tr>
<th>Order</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation 10</td>
<td>(Re(F_{p_z}))</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(Im(F_{p_z}))</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Simulation 11</td>
<td>(Re(F_{p_z}))</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(Im(F_{p_z}))</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Simulation 12</td>
<td>(Re(F_{p_z}))</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(Im(F_{p_z}))</td>
<td>-800</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Simulation 13</td>
<td>(Re(F_{p_z}))</td>
<td>800</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(Im(F_{p_z}))</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>Simulation 14</td>
<td>(Re(F_{p_z}))</td>
<td>-400</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(Im(F_{p_z}))</td>
<td>700</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Simulation 15</td>
<td>(Re(F_{p_z}))</td>
<td>0</td>
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<td></td>
<td>(Im(F_{p_z}))</td>
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</tr>
<tr>
<td>Simulation 16</td>
<td>(Re(F_{p_z}))</td>
<td>1200</td>
<td>1800</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(Im(F_{p_z}))</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Simulation 17</td>
<td>(Re(F_{p_z}))</td>
<td>0</td>
<td>800</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(Im(F_{p_z}))</td>
<td>-300</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Simulation 18</td>
<td>(Re(F_{p_z}))</td>
<td>200</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(Im(F_{p_z}))</td>
<td>0</td>
<td>700</td>
<td>-900</td>
</tr>
</tbody>
</table>

Results of the rotational simulations are presented in Table 4.3. From these results it can be seen that a numerical solution for the Fourier coefficients of Euler angle \(\Theta_{p_z}\) and angular velocity \(\Omega_{p_z}\) converges into analytical solution. Convergence of the Fourier coefficients can be seen in figures in appendix B.1. Additionally, comparison of the numerical and analytical solution for the Euler angle \(\theta_z\) and angular velocity \(\omega_z\) can be found in figures in appendix B.2.
Table 4.4: Results of the rotational validation simulations

| Order, $p$ | $|\Theta_{p_z}|$, rad | 1st     | 2nd     | 3rd     | 4th     |
|------------|---------------------|---------|---------|---------|---------|
| Simulation 10 | numerical           | 0.00012411 | 0 | 0 | 0 |
|              | analytical           | 0.00012411 | 0 | 0 | 0 |
| Simulation 11 | numerical           | 0.00012411 | 0 | 0 | 0 |
|              | analytical           | 0.00012411 | 0 | 0 | 0 |
| Simulation 12 | numerical           | 0.099294   | 0 | 0 | 0 |
|              | analytical           | 0.099294   | 0 | 0 | 0 |
| Simulation 13 | numerical           | 0.89126    | 0 | 0 | 0 |
|              | analytical           | 0.89126    | 0 | 0 | 0 |
| Simulation 14 | numerical           | 0.10006    | 0 | 0 | 0 |
|              | analytical           | 0.10006    | 0 | 0 | 0 |
| Simulation 15 | numerical           | 0.074471   | 0.018617 | 0 | 0 |
|              | analytical           | 0.074471   | 0.018617 | 0 | 0 |
| Simulation 16 | numerical           | 0.66845    | 0.33422  | 0 | 0 |
|              | analytical           | 0.66845    | 0.33422  | 0 | 0 |
| Simulation 17 | numerical           | 0.14894    | 0.055853 | 0 | 0 |
|              | analytical           | 0.14894    | 0.055853 | 0 | 0 |
| Simulation 18 | numerical           | 1.13690    | 1.00267  | 0 | 0 |
|              | analytical           | 1.13690    | 1.00267  | 0 | 0 |

4.6 Closure

In this chapter, a simple test case was designed and simulated in order to check the accuracy of the motion solution method displayed in the previous chapter. Variety of translational and rotational simulations are performed in order to cover a wide range of cases that are possible in
practice. The results of all simulations are given in form of tables and figures in both domains. The conclusion of all simulations is that motion algorithm gives results that are exactly the same as the analytical solution after two hundred to two hundred and fifty iterations. Next step is to perform simulations in which motion is induced by a surface wave.
5 Motion validation of a submerged body

5.1 Introduction

In this chapter heave motion of a body is compared to results of a transient simulation. Transient simulation results obtained using the present numerical framework are considered referent since they have been thoroughly verified and validated in numerous publications [2, 24, 25].

This chapter is divided in 5 sections. In the first section, the problems that have been experienced during the design of the test case are presented, which affects the final appearance of the case. Next, the set up that is used for HB simulation will be presented. Additionally, results of the corresponding transient simulation will be given. In the last section the results of the HB simulation and their comparison with the transient simulation results will be presented.

5.2 Design of the test case

The goal of this test is to compare motion calculated using Harmonic Balance with results of the corresponding transient simulation. Since this is the first simulation in which fluid flow and rigid body motion are coupled, a simple 2D test is designed. Rigid body in a shape of trapezoid was chosen because it best approximates a shape of a ship.

Trapezoidal body was placed on the free surface, however this simulation diverged. Since this was a 2D case, incoming air could not flow to the sides of the body, instead it was getting trapped under the bow. This accumulation of air eventually caused divergence of solution since this phenomena has a high frequency, which is not captured by the Fourier series expansion (see Eq. (2.14)), Instabilities on the free surface due to the air accumulation can be seen in Figure 5.1b, while the free surface at the beginning of the simulation can be seen in Figure 5.1a. Although the free surface is captured with LS method, for clarity the free surface is presented using $\alpha$ volume fraction. Note that $\alpha$ volume fraction will be used throughout this work to present the free surface.

![Image of free surface with $\alpha$ volume fraction]
In order to avoid the divergence of the simulation, the rigid body is submerged under the free surface. Hence, trapezoidal shape of the body was no longer suitable since its sharp edges would cause unwanted transient effects in the numerical solution. Therefore, a symmetrical rigid body in a shape of a hexagon is selected for further study. As it can be seen in Figure 5.2, the length of the rigid body is $L_b = 6\,\text{m}$, while its height is $h_b = 0.6\,\text{m}$.

Initially, rigid body is placed at a depth of $1.67\,\text{m}$. This depth will be used in the reference transient simulation, while for the reasons explained below the depth will be modified in the HB simulation. Positions of the upstream and downstream boundaries of the computational domain are selected in a way to avoid wave reflection. Hence, length of the computational domain upstream of the body is equal to two and a half lengths of the free surface wave, $2.5\lambda$, while the length of the domain in the downstream direction is equal to three and a half lengths of the free surface wave, $3.5\lambda$. Domain height is $6.71\,\text{m}$, where the water takes up 81.37% i.e. $5.46\,\text{m}$. Computational domain dimensions can be seen in Figure 5.3.
For the mass of the body that corresponds to the buoyancy force acting on the body, a non-periodic heave motion is obtained as it can be seen in Figure 5.4.

In order to use current HB motion algorithm which cannot calculate the mean value of motion, mean value changes in the transient simulation must be nullified. Nullification of the mean value changes in the transient simulation is achieved by using a linear spring and the body mass greater than the buoyancy force.

Since we use body mass that is greater than the buoyancy force, the body will sink until the equilibrium of buoyancy, gravitational and spring force is achieved. Furthermore, since mean value motion cannot be calculated with HB simulation, the value for which the body sank in the transient simulation must be taken into account beforehand i.e in the HB simulation body is placed deeper by the mean value motion from transient simulation. Mean value motion of the transient simulation is $0.492934 \text{m}$. Therefore, although overall
dimensions of the computational domain are the same in the case of the transient and HB simulation, the position of the body is different as it can be seen in Figure 5.5.

![Figure 5.5: Computational domain dimensions for the HB simulation.](image)

5.3 Numerical case set up

In this chapter set up for the HB simulation is presented in detail, and a brief review of the reference transient simulation is given.

5.3.1 FV mesh

FV mesh is generated using software Pointwise. The mesh is block structured with three longitudinal and six vertical blocks. All of 32718 cells are in the shape of hexahedra. In order to reduce the number of cells this mesh is graded in both longitudinal and vertical direction. In the longitudinal direction cells are graded towards the middle where object is placed, while in the vertical direction there are two distinct levels of mesh grading. First level of grading represents cells around body that are mildly graded, while the second level of cells grading is near the by free surface. Mesh is made with 10 elements per wave height. Full view of the FV mesh can be seen in Figure 5.6, while the zoomed detail of the mesh which shows the mesh around the body and near the free surface can be seen in Figure 5.7.

![Figure 5.6: Full view of the mesh.](image)
5.3.2 Relaxation zones

Relaxation zones are used to prevent wave reflection which can disrupt CFD results. Wave reflection occurs when the perturbation components do not vanish near the boundaries of the domain [16]. In relaxation zones, the solution is a linear combination of CFD and potential flow solution. It was shown by Vukčević et al [16] that when the length of the relaxation zone is greater than $\lambda$ amplitude of the reflected wave is below 1% of the incident wave amplitude. Therefore, the length of the relaxation zone at inlet and outlet, in this case, is set to one and a half wave length, $\lambda_r = 1.5\lambda$. Figure 5.8 shows weight field that represents relaxation zones. When the weight field is equal to zero, $w = 0$, the solution consists only of the CFD results, and when the weight field is equal to one, $w = 1$, only incident field comprises the solution.

5.3.3 Wave parameters

The dimensions of the computational domain and length of the relaxation zones are determined with the respect to the wave length of the free surface wave. In this case the wave length is $\lambda = 11.84\text{m}$, while the wave height is $h = 0.196\text{m}$. Angular frequency $\omega$ can be calculated with the following equation from linear wave theory:
where \( g = 9.81 \text{ m/s}^2 \) stands for gravitational acceleration. With the known angular frequency \( \omega \) we can easily calculate period \( T \) or wave frequency \( f_w \). However we must first take into the account the speed of the submerged body. The wave would affect the body motion with the frequency calculated with Eq. (5.1) in case of a stationary rigid body. Since the rigid body has the speed of \( U = 2 \text{ m/s} \) in the \(-x\) direction (see Figure 5.4), it will encounter waves at a higher frequency. It should be stated here that waves are propagating in the direction of the \( x \) axis. Therefore, encounter frequency is calculated by the following equation:

\[
\omega_e = \omega \left(1 + \frac{U \omega}{g}\right).
\] (5.2)

All relevant wave parameters are shown in Table 5.1.

| Wave height \( H, \text{ m} \) | 0.196 |
| Wave length \( \lambda, \text{ m} \) | 11.84 |
| Wave frequency \( \omega, \text{ rad/s} \) | 2.2816 |
| Speed of a body \( U, \text{ m/s} \) | 2 |
| Encounter frequency \( \omega_e, \text{ rad/s} \) | 3.343 |
| Encounter period \( T_e, \text{ s} \) | 1.8795 |
| Relaxation zone length \( \lambda_r, \text{ m} \) | 17.76 |

It should be noted that in our simulations rigid body does not move in the direction of \(-x\) axis, but we produce the same effect by inducing a steady current in the opposite direction.

### 5.3.4 Boundary conditions

Free surface and velocity are initialised using the nonlinear stream function wave theory [26] solution at given time instances. Therefore, boundary conditions are set up in a way that enables velocity and free surface initialisation with the wave theory, i.e. the perturbation components of velocity \( u_P \) and level set \( \psi_P \), as well as dynamic pressure are initialised to zero.

### 5.3.5 Simulation properties

**Phases.** Two phases used in this case are water and air. Density of the water is \( \rho_w = 1000 \text{ kg/m}^3 \), while its kinematic viscosity is \( v_w = 1 \cdot 10^{-6} \). Density of air in this case is set to \( \rho_a = 1 \text{ kg/m}^3 \), while its kinematic viscosity is set to \( v_a = 1.48 \cdot 10^{-5} \). The flow of both phases is laminar.

**Rigid body.** Body mass is set to \( m = 1000 \text{ kg} \), and stiffness coefficient of linear spring is set to \( c = 9000 \text{ Ns/m} \).
Numerical schemes. In order to obtain numerical solution for the governing equations that are described in second chapter (Chapter 2), FV discretization process must be carried out. Once the discretisation is carried out we can choose different numerical schemes for convection and diffusion terms [27].

Diffusion. For the diffusion terms of the perturbation components following numerical schemes are used: in the momentum equation second order, upwind-biased, unbounded scheme is used, while in the LS equation numerical scheme with van Leer limiter is used. For incident components second-order, unbounded schemes are used.

Convection. For the dynamic pressure gradient in the momentum equation linear interpolation is used with implicit jump conditions across the interface. Incident components of the velocity and LS field are discretised using linear interpolation.

5.4 Referent transient simulation

Transient simulation is performed using the same mesh as in the HB simulation. The only difference between HB and transient mesh is the body placement as it can be seen by comparing Figures 5.3 and 5.5. In order to ensure periodic convergence, 200 periods are simulated with 650 time-steps per wave period. First \(X_1\) heave motion order harmonic amplitude is obtained with Fast Fourier Transform (FFT) of the position signal in time. Analogously, first, \(F_1\) and second \(F_2\) order harmonic amplitude of force is obtained with FFT of force signal in time. The input of the FFT is the moving window with the width corresponding to the encounter period \(T_e\). Successive moving window FFT’s are used to determine the periodic convergence of the first and second order harmonic amplitudes.

Figure 5.9 presents the time series of heave motion, while the corresponding periodic convergence of first order of motion can be seen in Figure 5.10. Figure 5.11 presents the time series of heave force that acts on a body, while Figure 5.12 shows periodic convergence of first order force amplitude with respect to the number of simulated periods. Converged values of first and second order amplitudes of position and force that will be used for comparison with the HB simulation are given in Table 5.2.

Table 5.2: Periodically converged results for first and second harmonic amplitudes of the position and force.

<table>
<thead>
<tr>
<th>Order, (i)</th>
<th>position, (X_i) [m]</th>
<th>force, (F_i) [N]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>0.01308</td>
<td>28.6312</td>
</tr>
<tr>
<td>2nd</td>
<td>0</td>
<td>1.07</td>
</tr>
</tbody>
</table>
Figure 5.9: Time signal of the rigid body position.

Figure 5.10: Periodic convergence of the position Fourier coefficients.
5.5 Results

Results of the HB simulation that are presented in this section are obtained with six harmonics. Initially a simulation with two harmonics was performed, however it diverged which is why the number of harmonics was increased. With four harmonics the simulation converged, however
in order to establish sensitivity of the result to the number of used harmonics, an additional simulation with six harmonics is performed. Since the difference between results obtained from simulations with four and six harmonics is around 1%, additional increase in the number of harmonics was rendered unnecessary. In order to ensure convergence 20000 iterations are performed.

First and second order harmonic amplitudes of position and force are given in Table 5.3, while the convergence of first order position and first order force amplitude is shown on Figure 5.13 and Figure 5.14, respectively.

**Table 5.3: First and second order harmonic amplitude of the position and force obtained from HB simulation with 6 harmonics.**

<table>
<thead>
<tr>
<th>Order, $i$</th>
<th>motion, $X_i$ [m]</th>
<th>force, $F_i$ [N]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>0.01468</td>
<td>31.8 ± 0.6</td>
</tr>
<tr>
<td>2nd</td>
<td>0</td>
<td>1.35 ± 0.35</td>
</tr>
</tbody>
</table>

![Figure 5.13: Convergence of the first order position amplitude.](image)
Table 5.4 shows the results of the first order harmonic amplitude of position and the first and second order harmonic amplitudes of force for different number of harmonics and relative difference with respect to the transient simulation. The absolute value of relative difference $\varepsilon = (S_t - S_{hb})/S_t$ between HB and transient simulation is given in percentages, where $S_t$ denotes transient solution and $S_{hb}$ denotes the HB solution. The relative difference of the first order position amplitude for the simulation with four harmonics is 13.6%, and for the simulation with six harmonics relative difference drops to 12.23%.

The first and second order force amplitudes in the HB simulation are not fully converged, i.e. they are periodically changing between minimum and maximum values. Hence, in Table 5.4 values for force amplitudes are given as a range of values. In the simulation with four harmonics, biggest relative difference of first order force amplitude is 15.96%, while in the simulation with six harmonics biggest relative difference drops to 13.16%. Since the value of the second order force amplitude is changing nearly $\pm 50\%$ from its mean value in both simulations they cannot be taken into consideration. Nevertheless, because the magnitude of the second order force amplitude is relatively small with respect to the magnitude of first order force amplitude it does not affect the overall motion of the body. Note that the second order amplitude of position is not given in Table 5.4 since it is negligibly small.

Figure 5.14: Convergence of the first order force amplitude.
Table 5.4: Results of the HB simulations with 4 and 6 harmonics and the relative difference towards transient simulation.

| No. Harmonics | $X_1$ [m] | $F_1$ [N]    | $F_2$ [N]    | $|\varepsilon_{X_1}|$, % | $|\varepsilon_{F_1}|$, % | $|\varepsilon_{F_2}|$, % |
|----------------|----------|--------------|--------------|----------------|----------------|----------------|
| 4              | 0.01486  | 32 ± 1.2     | 1.3 ± 0.6    | 13.6           | 11.765 ± 4.195 | 77.57          |
| 6              | 0.01468  | 31.8 ± 0.6   | 1.35 ± 0.35  | 12.23          | 11.065 ± 2.095 | 58.87          |

The Figures 5.15, 5.16, 5.17 and 5.18 show perturbation velocity fields $u_P$ at randomly selected time instants and their corresponding field from transient simulation.

From the figures it can be seen that the velocity fields in water are similar. However, the magnitude of the perturbation velocity field in the air is greater in the transient simulation, while the periodic distribution of the velocity is more representative in the HB simulation. This is very likely related to the existence of high values of dynamic pressure in the relaxation zones of transient simulation, as can be seen in Figures 5.19, 5.20 and 5.21. Nevertheless, from the same figures it can be seen that the dynamic pressure fields around the submerged body are similar. Perturbation velocity field differences in air do not affect the motion in this case since the body is submerged.
Figure 5.16: (a) HB simulation at $3T/13$, (b) Transient simulation at $3T/13$.

Figure 5.17: (a) HB simulation at $5T/13$, (b) Transient simulation at $5T/13$. 
Figure 5.18: (a) HB simulation at $8T/13$, (b) Transient simulation at $8T/13$.

Figures 5.19, 5.20 and 5.21 are showing comparable dynamic pressure fields at randomly selected time instants.

Figure 5.19: (a) HB simulation at $6T/13$, (b) Transient simulation at $6T/13$.

Figure 5.20: (a) HB simulation at $9T/13$, (b) Transient simulation at $9T/13$. 
5.6 Conclusion

Goal of this chapter was to validate heave motion of a rigid body calculated with HB simulation. It is shown that relative difference between HB and the corresponding transient simulation is between 10% and 15%. The discrepancies are acceptable since the motion of the submerged body is extremely mild. First order amplitude of force is only 0.32% of the body weight, while the amplitude of motion is smaller than 10% of wave height.

In the future work it is necessary to validate rotational motion and the complete 6DOF motion with a 3D case. Nevertheless, results in this chapter are confirming that the HB method can be used to simulate wave induced motion.
6 Conclusion and Future Work

This thesis presents a method for rigid body motion calculation which is compatible with the HB method that is used to calculate flow field. New method is tested on two test cases. In the first test case, translational and rotational motion is validated with analytical results, while in the second test case results are compared to the corresponding transient simulation.

In order to be able to describe the derivation of governing equations for rigid body dynamic in the frequency domain, existing HB method which is used to simulate two-phase, nonlinear and viscous flow is explained. First, governing equations in the basic form and the jump conditions at the interface are described, followed by the SWENSE decomposition method and governing equations in SWENSE form. Next, basics of HB method are given as well as the advantages of the implicit coupling of the source term. At last, HB method is used to transform SWENSE decomposed governing equation into time-spectral form, i.e. HB form. These equations are then implemented in a numerical solver to simulate two-phase, nonlinear and viscous flow. Following the brief overview of HB method, detailed derivation of the motion equations is given.

Validation of the new method is then conducted on two test cases. Goal of the first test case is to validate translational and rotational motion when the force and torque remained constant during the simulation. In order to check the accuracy of the method, complexity of forces and torques acting on the body increased up until the point when they were given as a combination of four orders of harmonic amplitudes. Results for this test case showed that this method can be used since numerical and analytical results are the same.

Second test case was designed to validate wave-induced body motion. In this case only heave motion was tested. Results of the HB simulation are compared to transient simulation results. Results show around 12% of relative difference for position and around 15% relative difference for force. These results are good enough considering the mild motion of the body. Hence, the results have shown that the body motion can be calculated in frequency domain.

Disadvantage of the method that is now implemented is the inability to calculate mean value motion. That should be in the focus of future work. Furthermore, in this thesis only translational motion induced by waves was validated, so in future work rotational motion should be validated. Additionally, the final validation study should be performed on a realistic 3D ship.

In short, this thesis shows that ship motion calculation in naval hydrodynamics can be performed in the frequency domain with the HB method. With additional work this method should lead toward automatic optimisation of ship resistance in waves in the process of ship design.
References


A Translational motion solution

A.1 Frequency domain

Figure A.1: Convergence of Fourier coefficients the in the 1st and 2nd simulation.
Figure A.2: Convergence of Fourier coefficients in the 3rd and 4th simulation.

Figure A.3: Convergence of Fourier coefficients in the 5th simulation.
Figure A.4: Convergence of Fourier coefficients in the 6th simulation.
Figure A.5: Convergence of Fourier coefficients in the 7th simulation.

Figure A.6: Convergence of Fourier coefficients in the 8th simulation.
Figure A.7: Convergence of Fourier coefficients in the 9th simulation.

A.2 Time domain
b) Velocity,

Figure A.8: Time domain variables during one period in the 1st simulation.

a) Position,

b) Velocity,

Figure A.9: Time domain variables during one period in the 2nd simulation.
Figure A.10: Time domain variables during one period in the 3rd simulation.
b) Velocity,

Figure A.11: Time domain variables during one period in the 4th simulation.

a) Position.

b) Velocity.

Figure A.12: Time domain variables during one period in the 5th simulation.
Figure A.13: Time domain variables during one period in the 6th simulation.
b) Velocity,

Figure A.14: Time domain variables during one period in the 7th simulation.

a) Position,

b) Velocity,

Figure A.15: Time domain variables during one period in the 8th simulation.
Figure A.16: Time domain variable during one period in the 9th simulation.
B  Rotational motion solution

B.1  Frequency domain

Figure B.1: Convergence of Fourier coefficients in the 10th and 11th simulation.
b) Angular velocity,

Figure B.2: Convergence of Fourier coefficients in the 12th and 13th simulation.

a) Euler angle,

Figure B.3: Convergence of Fourier coefficients in the 14th simulation.
Figure B.4: Convergence of Fourier coefficients in the 15th simulation.
Figure B.5: Convergence of Fourier coefficients in the 16th simulation.

Figure B.6: Convergence of Fourier coefficients in the 17th simulation.
Figure B.7: Convergence of Fourier coefficients in the 18th simulation.

B.2 Time domain
b) Angular velocity,

Figure B.8: Time domain variables during one period in the 10th simulation.

a) Euler angle,

b) Angular velocity,

Figure B.9: Time domain variables during one period in the 11th simulation.
Figure B.10: Time domain variables during one period in the 12th simulation.
b) Angular velocity,

Figure B.11: Time domain variables during one period in the 13th simulation.

a) Euler angle,

b) Angular velocity,

Figure B.12: Time domain variables during one period in the 14th simulation.
Figure B.13: Time domain variables during one period in the 15th simulation.
b) Angular velocity,

Figure B.14: Time domain variables during one period in the 16th simulation.

b) Angular velocity,

Figure B.15: Time domain variables during one period in the 17th simulation.
a) Euler angle,

b) Angular velocity,

Figure B.16: Time domain variables during one period in the 18th simulation.